

# COMPETITIVE BARGAINING EQUILIBRIUM

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**ABSTRACT.** We propose a simple bargaining procedure, the equilibrium of which converges to the Walrasian allocation as the agents become increasingly patient. We thus establish that the competitive outcome obtains even if agents have market power and are not price-takers. Moreover, where in other bargaining protocols the final outcome depends on bargaining power or relative impatience, the outcome here is determinate and depends only on preferences and endowments. This procedure has therefore important implications for policy applications compared to standard bargaining rules.

*Keywords.* Bargaining. Walrasian Equilibrium. Price-setting. Quantity Constraints. Search. Matching. *JEL.* C60. C71. D41. D51.

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## 1. INTRODUCTION

Price-taking behavior is typically invoked as a necessary requirement to obtain the competitive outcome. In this paper, we propose a bargaining foundation for the Walrasian equilibrium in a small exchange economy where agents are not price-takers. The bargaining procedure we analyze is a variation of the standard alternating-offers bargaining: each agent alternately sets a price and a constraint on the quantities to be exchanged, and the respondent either accepts and chooses the quantities to be traded at the terms of the offer, or rejects and makes an offer in the next period in which utilities are discounted. We show that competition and the competitive outcome is obtained even without price-taking behavior when bargaining frictions vanish. As a result, the outcome does not depend on details such as relative impatience and bargaining power or outside options. Paradoxically, by explicitly introducing price-setting as a strategic variable in an otherwise standard bargaining environment, the competitive outcome is restored. Price-taking is therefore not a necessary requirement for attaining the perfectly competitive outcome.

Prices have of course long been recognized as the key driving force in the efficient functioning of markets. In markets with a large number of buyers and sellers, and provided each of them is small relative to the size of the market, all agents make their consumption and production decisions taking the price determination process as given. While in reality sellers very often set prices (from high street clothing items to supermarket perishables), a seller has no incentive to price his product above the market price as she will not attract any buyers. This pricing behavior is therefore accurately modeled by means of a fictitious Walrasian auctioneer. Nevertheless, in many real world markets, some or all agents are effectively price-setters instead, either because the number of players in the market is small, or because some players are large relative to the market. In an exchange economy with two agents for example, the determination of prices through price-setting is not adequately represented by a Walrasian auctioneer as each of them has market power.

Rational agents in a two-sided negotiation will nonetheless agree to obtain an efficient allocation even when they have some market power. Edgeworth (1882) argues that, depending on the details of the contract between bargaining agents, any outcome in the set of Pareto-improving efficient allocations (the relevant portion of the contract curve) is attainable. Given this indeterminacy, standard bargaining theories have suggested particular outcomes to be selected on the contract curve. Typically, the outcome that is selected depends on the specifics of the bargaining procedure. For example, in the case of Nash bargaining, the outcome depends on the exogenously determined relative bargaining power of each of the agents and on the outside options chosen. In the case of alternating-offers bargaining (Rubinstein (1982) and Ståhl (1972)), the final outcome depends on the relative impatience of the bargaining parties. The relatively patient agent will obtain a proportionally larger share of the surplus.<sup>1</sup> The important implication is that because the outcome

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<sup>1</sup>Even in the case when *both* agents become infinitely patient, the rate at which they do so will determine the final outcome. Binmore, Rubinstein and Wolinsky (1986) establish that the alternating-offers bargaining solution approaches the Nash bargaining solution with utilities that reflect the incentives to settle and with the disagreement point properly chosen.

depends on the specifics of the bargaining procedure, the outcome is not in general the competitive equilibrium allocation.

In contrast with previous bargaining models, the bargaining procedure we propose will *always* converge to a competitive equilibrium allocation, and is therefore independent of anything other than the preferences and endowments of the agents. The outcome is completely determined by the primitives of the economy and does not depend on whether one agent becomes patient at a faster rate than the other agent, or whether one agent has more bargaining power than the other. This determinacy result delivering the Walrasian outcome is paradoxically driven by the role played by price-setting in the bargaining. More specifically, the agents alternate in offering a price with a maximum quantity constraint, while the other agent chooses the quantity traded if she accepts the offer. It is precisely this separation of the price-setting by the proposer from the quantity decision by the recipient that is necessary to guarantee the convergence to the Walrasian allocation. This price-quantity separation is obviously well known in negotiation<sup>2</sup> and has common applications in several economic environments such as union-wage bargaining in the labor economics literature,<sup>3</sup> and standard buy-out provisions in two-person partnerships.<sup>4</sup>

Because bargaining ends when an offer is accepted and the quantities to be traded are chosen, subgame perfection implies that any accepted offer must either be on the offer curve of the accepting agent,<sup>5</sup> or it must be equal to the quantity constraint announced in the offer. Of course, an agent can also reject the offer and make a counter-offer, which if accepted will lead to an allocation on the offer curve (or at the quantity constraint) of the accepting agent. Intuitively, when discount frictions are vanishing, agents must obtain the same utility from accepting an offer on their own offer curve as from making an offer that is accepted on the other agent's offer curve. One such allocation is obviously a Walrasian one, which is an intersection of both offer curves. Nevertheless, as will be shown below, a necessary requirement to guarantee convergence to the Walrasian allocation is that offers include the *quantity constraint* as well.

In effect, in the absence of the quantity constraint, convergence to the Walrasian allocation is not guaranteed.<sup>6</sup> Generically, there will exist at least one other equilibrium that does *not* converge to the Walrasian allocation but instead to another, inefficient allocation. At any such equilibrium, each agent obtains the same utility

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<sup>2</sup>The classic example of separating the "general" terms of an agreement from the actual choice of the outcome is the division of a cake by two children: one cuts the cake and the other one chooses first which piece to eat.

<sup>3</sup>See Solow and MacDonald (1981) and Farber (1986) among others who study and document such bargaining over wages where the union negotiates the wage and the employer chooses the level of employment.

<sup>4</sup>Cramton, Gibbons and Klemperer (1987), and Moldovanu (2001) model such buy-out provisions. When partners decide they want to separate, the provision prescribes that one partner chooses the price of the shares, and the other partner chooses the quantity traded, i.e. whether to buy or sell.

<sup>5</sup>By definition, a point on the offer curve of an agent within the Edgeworth box is an allocation that maximizes her utility at a given relative price.

<sup>6</sup>The first paper to explore a bargaining procedure with price offers is Yildiz (2002). While he finds convergence to the Walrasian equilibrium, in a previous working paper (Dávila and Eeckhout (2002)) we show that such convergence obtains only under very special, non-generic conditions.

from consuming an allocation on her own offer curve and a different allocation on the offer curve of the opponent corresponding to a different price. What the quantity constraint does in the bargaining procedure proposed here is to make sure that one of the agents has an incentive not to stick to an offer that produces inefficient allocations. She can get away from the inefficient allocation by offering a Pareto efficient allocation that makes her strictly better off and the other agent not worse off. The way to obtain that efficient allocation is to restrict the quantities traded to be smaller than the trades desired by the other agent at the newly offered price.

Our bargaining procedure captures the main aspects of several existing price setting mechanisms. For example, in commodity futures markets,<sup>7</sup> the seller of future contracts will typically announce to a candidate buyer the price for the contract and how many contracts he has on offer. The candidate buyer can accept the price offer and choose the number of future contracts as long as it does not exceed the quantity constraint that was offered initially. The same is true for limit orders when selling stock. Your limit order guarantees a certain price for the stock, but you cannot be sure that the order will be filled. Only if there is sufficient demand at that price will your order be filled (either partially or completely). Another price setting mechanism that corresponds to the one in our model is that of interest rate quotations on mortgages. Mortgage lenders typically will offer a rate that is valid for mortgage amounts up to a maximum. The home buyer can then choose any mortgage amount below that maximum.

This bargaining procedure, like Nash or alternating-offers bargaining, proposes a solution to the Edgeworth indeterminacy result by selecting a particular outcome on the contract curve. However, there is one main difference. In the existing theories, the indeterminacy does not really disappear in the sense that the bargaining outcome is a function of a new, exogenously imposed property of the bargaining procedure (either bargaining power or relative impatience). In other words, Edgeworth's indeterminacy still remains, albeit now parameterized by an exogenous parameter. Our theory derives by means of a specific bargaining protocol a solution that is no longer dependent on any exogenous parameter but only on the primitives of the economy (preferences and endowments).

The independence of the outcome on an exogenous parameter is crucial for the application of bargaining procedures in economic theory, for example in equilibrium search models (for an overview, see Mortensen and Pissarides (1999)). Typically, due to thick market externalities, search markets are inefficient.<sup>8</sup> Hosios (1990) derives a simple condition that needs to hold in order to obtain efficiency. Unfortunately, that condition depends on the specifics of the bargaining procedure (i.e. the Nash bargaining share if the surplus is divided using Nash bargaining). The policy prescription therefore depends on one additional parameter. Moreover, information about that parameter is often hard to obtain, and without knowledge about that parameter, the policy prescription is indeterminate. In the case the surplus in a pairwise meeting is split using our bargaining procedure, Hosios' policy prescription for efficiency will be completely determinate, independently of bargaining power for example.

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<sup>7</sup>We are grateful to Dale Mortensen for pointing this out to us.

<sup>8</sup>Moen (1996) and Acemoglu and Shimer (1999a and b) have proposed theories of search that are efficient when atomistic firms in a large market announce wages.

Finally, Gale (1986) establishes a bargaining foundation for the Walrasian equilibrium outcome in general exchange economies with a continuum of traders and random pairwise matching. While our setup is completely different, our results contribute to this literature by extending the bargaining foundation for Walrasian equilibrium even for economies with a small number of agents. Despite the (only two) agents being price-setters, the perfectly competitive outcome still obtains.

In the next section, we lay out the model. In section 3, we present the main result of convergence to the Walrasian allocation and we illustrate how we obtain it. Formal proofs are relegated to the Appendix. The results are derived for equilibria in which there is immediate acceptance. In section 4 we study equilibria with delay and prove that no such equilibria exist. We then discuss in section 5 the results in relation to the existing literature and we analyze the differences with other bargaining procedures: bargaining over allocations, Nash bargaining, bargaining over prices only, bargaining with minimum (instead of maximum) quantity constraints, and bargaining when there is a fixed cost of delay instead of discounting. Finally, section 6 finishes with some concluding remarks.

## 2. THE MODEL

Consider an exchange economy with two agents  $A$  and  $B$ , each endowed with two goods 1 and 2 over which they have preferences represented by standard<sup>9</sup> utility functions  $u^A$  and  $u^B$ . Let  $e^A = (e_1^A, e_2^A)$  and  $e^B = (e_1^B, e_2^B)$  be the endowments of agents  $A$  and  $B$  respectively, and let  $e = e^A + e^B$  denote the total endowments. The vector of goods consumed by  $A$  and  $B$  is given by  $x^A = (x_1^A, x_2^A)$  and  $x^B = (x_1^B, x_2^B)$  respectively.

In general, for given endowments, the allocation is not Pareto efficient and there will exist gains from trade. The central issue in this paper is to address the question how those gains from trade are realized. More specifically, we are interested in establishing whether all gains from trade are completely achieved (i.e. whether the outcome is efficient) and if so, which of all efficient allocations is obtained.

If both agents are considered to be price-takers, we can solve for an allocation that exploits all gains from trade by invoking the centralized Walrasian auctioneer. This auctioneer "determines" a relative price  $p$  representing the terms of trade of, for instance, good 1 in terms of good 2 (that is, good 2 is treated as the numeraire good, with its price normalized to one). A Walrasian equilibrium allocation is any feasible allocation of consumption bundles  $\bar{x}^A, \bar{x}^B$  that maximize each agent's utility given a price  $\bar{p}$  announced by the Walrasian auctioneer. The first theorem of welfare economics establishes the efficiency of the Walrasian equilibrium.

Nonetheless, since the price-taking assumption is not easily justified in a two-person economy, we propose a bargaining procedure that allows agents to realize the gains from trade and in which agents set prices. More specifically, we consider an alternating-offers bargaining game in which, in any given period, one of the

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<sup>9</sup>That is to say, continuous in  $\mathbb{R}_+^2$ , differentiable in  $\mathbb{R}_{++}^2$ , monotonous in the sense that  $Du(x) \in \mathbb{R}_{++}^2$  always, quasi-concave in the sense that  $D^2u(x)$  is negative semi-definite in the normal space of  $Du(x)$  always, and well-behaved in the boundary in the sense that  $u^{-1}(a) \subset \mathbb{R}_{++}^2$  for any  $a \in u(\mathbb{R}_{++}^2)$ .

agents offers to the other a relative price at which he is willing to trade up to some maximum amount (for the remainder of the paper, we will refer to this as the *quantity constraint*).<sup>10</sup> Thus an offer by  $A$ , for instance, consists of a relative price  $p^A$  (of good 1 in terms of good 2) and a quantity constraint  $q^A$  on the trade of good 1.<sup>11</sup> Represented in an Edgeworth box, the offer  $(p^A, q^A)$  corresponds to a linear segment starting from the endowments point  $e$  with slope  $p^A$  and a first coordinate with absolute value  $q^A$ . Upon the reception of an offer, the recipient, i.e.  $B$  in this case, can either accept the offer or reject it. If she accepts, she then chooses her most preferred consumption  $\tilde{x}^B(p^A, q^A) = (\tilde{x}_1^B(p^A, q^A), \tilde{x}_2^B(p^A, q^A))$  (and hence her desired trade) at the offered price, without violating the quantity constraint expressed in the offer. That is to say, she chooses an  $\tilde{x}^B(p^A, q^A)$  that is a solution to<sup>12</sup>

$$\begin{aligned} & \max u^B(x^B) \\ \text{s.t. } & p^A(x^B - e^B) \leq 0 \\ & |x_1^B - e_1^B| \leq q^A. \end{aligned} \tag{1}$$

If  $B$  rejects the offer,  $B$  then counter-offers another pair  $(p^B, q^B)$  with a new relative price and a new quantity constraint. The utility of both agents  $A$  and  $B$  is discounted for every iteration of the bargaining by positive discount factors  $\delta^A$  and  $\delta^B$  not bigger than 1. Not reaching an agreement entails a zero utility to both agents.

A stationary subgame perfect (SSP) equilibrium with immediate acceptance consists of a pair of offers  $(p^A, q^A)$  and  $(p^B, q^B)$  such that, in every subgame where  $A$  is called to make an offer,  $A$  offers  $(p^A, q^A)$  and this offer solves

$$\begin{aligned} & \max u^A(e - \tilde{x}^B(p^A, q^A)) \\ \text{s.t. } & u^B(\tilde{x}^B(p^A, q^A)) \geq \delta^B u^B(e - \tilde{x}^A(p^B, q^B)) \end{aligned} \tag{2}$$

given  $(p^B, q^B)$ , and similarly for  $(p^B, q^B)$  given  $(p^A, q^A)$ .

In effect, from subgame perfection, once  $B$  decides to accept  $A$ 's offer, she will choose the consumption bundle  $\tilde{x}^B(p^A, q^B)$  that maximizes her utility subject to the terms of the offer. Therefore, knowing that upon acceptance  $B$  chooses  $\tilde{x}^B(p^A, q^B)$ ,  $A$  decides to make an offer that maximizes his utility of consuming  $e - \tilde{x}^B(p^A, q^B)$ , provided that the offer induces  $B$  to accept it. This requires that  $B$  obtains at least as much utility from accepting the current offer, i.e.  $u^B(\tilde{x}^B(p^A, q^B))$ , as she would get from rejecting the offer and waiting for her equilibrium offer  $(p^B, q^B)$  to be accepted in the next period, which gives her a utility  $\delta^B u^B(e - \tilde{x}^A(p^B, q^B))$ .

It turns out to be the case that there is no loss of generality in focusing on the SSP equilibria with immediate acceptance. As a matter of fact, in this paper we

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<sup>10</sup>To be precise, we refer to a *maximum* quantity constraint. We will discuss the case of a *minimum* quantity constraint below in section 5.

<sup>11</sup>For notational convenience, we have chosen to model the quantity constraint by means of the first coordinate of the vector of trades. This does not suppose any loss of generality as we could as well have chosen to model it as a maximum restriction on the modulus of the trade, for example, without any impact whatsoever on the results.

<sup>12</sup>Abusing the notation only slightly we let the relative price  $p^A$  stand for the normalized vector of prices  $(p^A, 1)$ , and similarly for  $p^B$ .

will consider *all* types of SSP equilibria, i.e. whether with immediate acceptance as defined above by conditions (1) and (2), or with delay. First, we concentrate in section 3 on equilibria with immediate acceptance, and then we show in section 4 that there are no SSP equilibria with delay whenever agents are impatient.

In the next section we present the result establishing that the outcome of bargaining over prices and quantity constraints converges to a Walrasian allocation as the agents become infinitely patient. The proofs are provided in the appendix.

### 3. THE CONVERGENCE RESULT

In this section, we develop the argument that establishes the convergence to a Walrasian allocation of every SSP equilibrium with immediate acceptance of bargaining over prices and quantity constraints. The key insight is that in exchanging price and quantity offers, the agents are actually bargaining over *some* allocations. Each offer by say player  $A$  of a price and quantity constraint  $(p^A, q^A)$  corresponds to an allocation, in case  $B$  accepts the offer. Therefore, in a SSP equilibrium with immediate acceptance, agents *de facto* offer allocations, although they are restricted in the allocations they can offer by the optimal acceptance behavior of the receiver.

In effect, given subgame perfection of the equilibrium, any agent making an offer anticipates the optimal acceptance behavior by the recipient, and therefore an offer  $(p^A, q^A)$  amounts to offering the allocation  $(e - \tilde{x}^B(p^A, q^A), \tilde{x}^B(p^A, q^A))$ , where  $\tilde{x}^B(p^A, q^A)$  is the consumption chosen by  $B$  given the relative price  $p^A$  and the quantity constraint  $q^A$ .

This allows to characterize easily the consumption bundles that might be accepted at a SSP equilibrium with immediate acceptance. Note first that in the absence of a quantity constraint  $q^A$  (or equivalently, when the constraint is sufficiently slack),  $B$ 's response to  $A$ 's offer is to choose the allocation on her offer curve (i.e. her demand denoted by  $x^B(p^A)$ ) which maximizes  $B$ 's utility at price  $p^A$ . Note also that by means of the quantity constraint  $q^A$ , agent  $A$  can prevent agent  $B$  from attaining her demand  $x^B(p^A)$ , forcing her to trade less than in the unconstrained case. Nevertheless, in no instance can  $A$  force  $B$  to exchange more than necessary to attain her desired demand  $x^B(p^A)$  at that price. As a consequence, an offer by  $A$  that is responded with an optimal acceptance decision by  $B$  will always result in an allocation  $(e - x^B, x^B)$  such that

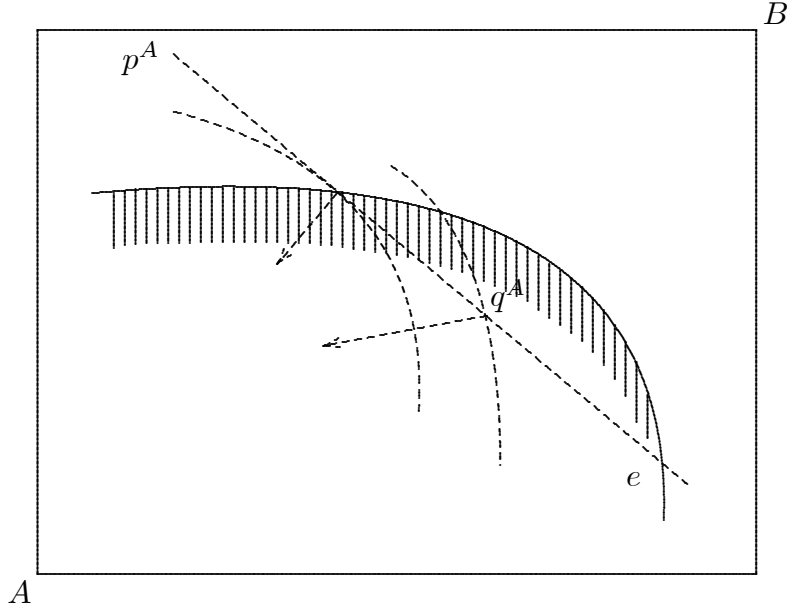
$$Du^B(x^B)(x^B - e^B) \geq 0, \quad (3)$$

where  $Du^B(x^B)$  is the gradient of  $u^B$  at  $x^B$ .

Condition (3) characterizes necessarily the solution to the maximization problem (1) above, and holds with equality if the maximum exchange constraint does not effectively constrain  $B$ 's choice and with strict inequality otherwise. This is illustrated in the Edgeworth box in Figure 1 below. Starting from the endowment  $e$ , the solid curve is  $B$ 's offer curve. For any price  $p^A$ , if unconstrained by  $q^A$ ,  $B$  will choose the allocation  $x^B(p^A)$  that maximizes her utility  $u^B$ , i.e. where the price schedule is tangent to her indifference curve. But  $A$ 's ability to constrain  $B$ 's demand allows  $A$  to make offers that will eventually lead  $B$  to accept in a bigger set of allocations, namely anything that is within the area delimited by  $B$ 's offer

curve (say "inside"  $B$ 's offer curve as opposed to being "on"  $B$ 's offer curve). The same is true for offers made by  $B$  with respect to  $A$ 's offer curve.

Figure 1



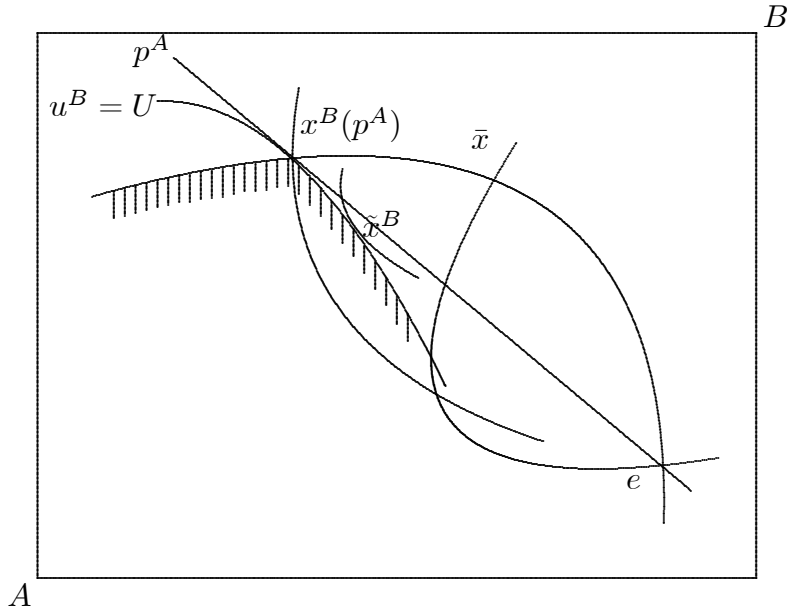
A SSP equilibrium with immediate acceptance of the bargaining over prices and maximum trades can then be characterized as a bundle  $x^B$  offered by  $A$  to  $B$  and a bundle  $x^A$  offered by  $B$  to  $A$  such that  $x^B$  solves

$$\begin{aligned} \max u^A(e - x^B) \\ Du^B(x^B)(x^B - e^B) \geq 0 \\ u^B(x^B) \geq \delta^B u^B(e - x^A) \end{aligned} \quad (4)$$

given  $x^A$ , and likewise for the offer  $x^A$  made by  $B$ , given  $x^B$ . Condition (4) is thus equivalent to conditions (1) and (2).

Now in order to see the convergence to a walrasian allocations of the SSP equilibria with immediate acceptance consider first the case of infinitely patient agents, i.e. with discount factors  $\delta^A, \delta^B = 1$ . Fix any utility level (not necessarily corresponding to a SSP equilibrium with immediate acceptance) that agent  $B$  can obtain from delaying one period at  $u^B = U$ . Then the problem (4) for agent  $A$  reduces to choosing  $x^B$  that maximizes  $A$ 's utility conditional on  $B$  accepting. This problem is illustrated graphically in Figure 2 for the case in which  $u^B = U$  is bigger than  $B$ 's utility  $u^B(\bar{x}^B)$  at the Walrasian allocation. The curves going through  $e$  in figure 2 represent the offer curves, the intersection of which is the Walrasian equilibrium allocation  $\bar{x}$ . Since any offer  $(p^A, q^A)$  made by  $A$  actually induces  $B$  to accept an allocation "inside"  $B$ 's offer curve and  $B$  must at least get  $u^B = U$ , the set of feasible allocations therefore corresponds to the shaded area.

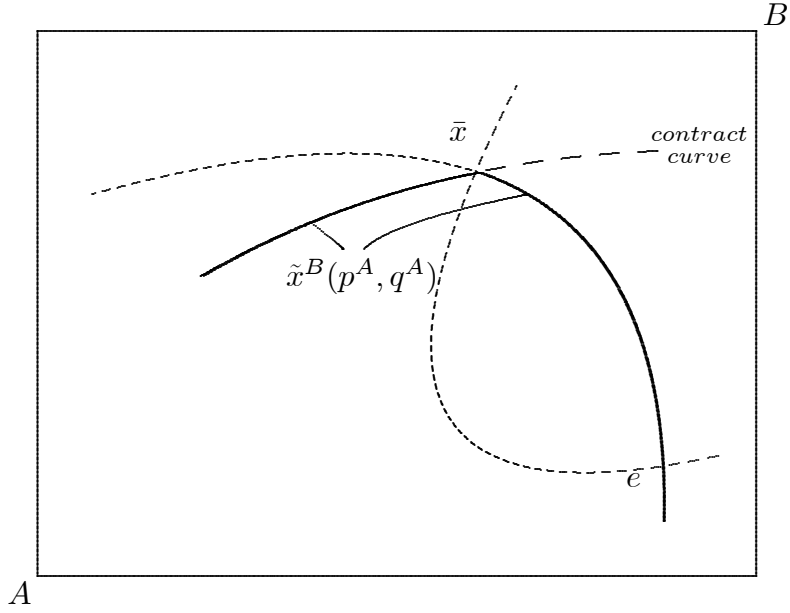
Figure 2



Now suppose that  $A$  offers a price  $p^A$  as given in Figure 2, and a slack quantity constraint (i.e.  $q^A$  large) implying acceptance by  $B$  at  $x^B(p^A)$  and giving  $B$  a utility of  $u^B = U$ . This is clearly not the solution to (4) for  $A$ , as he can offer a flatter price schedule and a binding quantity constraint such that his utility increases while keeping  $B$ 's utility at  $u^B = U$ . The solution is rather to make an offer  $(p^A, q^A)$  leading to the allocation  $\tilde{x}^B$ . It is immediate that for any such  $u^B = U$  the optimal solution requires the tangency of  $A$ 's indifference curve to  $B$ 's, which of course implies that it is on the contract curve. It is equally easy to see that for levels of utility  $u^B = U$  lower than  $u^B(\bar{x}^B)$ ,  $A$ 's optimal offer leads to a point on  $B$ 's offer curve.

Therefore, there will be solutions  $\tilde{x}^B$  parameterized by levels of  $u^B = U$  that are either on the contract curve, or on  $B$ 's offer curve, as shown in Figure 3 below. The former is the case when the contract curve is "within"  $B$ 's offer curve, as is the case in Figure 3 to the "left" of the Walrasian allocation, where  $u^B = U$  is larger than  $B$ 's utility at the Walrasian allocation  $u^B(\bar{x}^B)$ . In the latter case,  $A$  cannot induce  $B$  to accept an offer beyond  $B$ 's optimal acceptance decision on her offer curve. We will refer to the curve  $\tilde{x}^B$ , parameterized by  $u^B = U$ , as  $B$ 's modified offer curve.

Figure 3



The previous discussion is formalized by the following two auxiliary lemmas. Lemma 1 establishes first that condition (4) implies that at a SSP equilibrium with immediate acceptance, no agent leaves to the other agent more utility than what is strictly necessary to obtain his acceptance, therefore extracting all the rents. Note that this is not unlike the standard Rubinstein alternating-offers bargaining solution. Thus, any equilibrium offer will make the recipient exactly indifferent between accepting it and waiting one more period to have his counteroffer accepted.

**Lemma 1.** *At every SSP equilibrium with immediate acceptance  $p^A, q^A, p^B, q^B$ ,  $A$  leaves  $B$  not more utility than is strictly necessary to obtain his acceptance, that is to say*

$$u^B(\tilde{x}^B(p^A, q^A)) = \delta^B u^B(e - \tilde{x}^A(p^B, q^B))$$

and hence, from  $A$ 's problem's first order conditions

$$[Du^A(e - \tilde{x}^B(p^A, q^A)) - \lambda^A Du^B(\tilde{x}^B(p^A, q^A))] D\tilde{x}^B(p^A, q^A) = 0$$

for some  $\lambda^A \geq 0$ . Likewise for  $B$ .

Lemma 2 shows by means of Lemma 1 that in a SSP equilibrium with immediate acceptance, the allocation resulting from the acceptance must be on the offer curve of the accepting party if inefficient, while any offer that effectively constrains the demand of the accepting party has to lead to an efficient allocation.

**Lemma 2.** *For every SSP equilibrium with immediate acceptance  $p^A, q^A, p^B, q^B$ , if the offer accepted by  $A$  is not on his offer curve, then it is efficient. Likewise for  $B$ .*

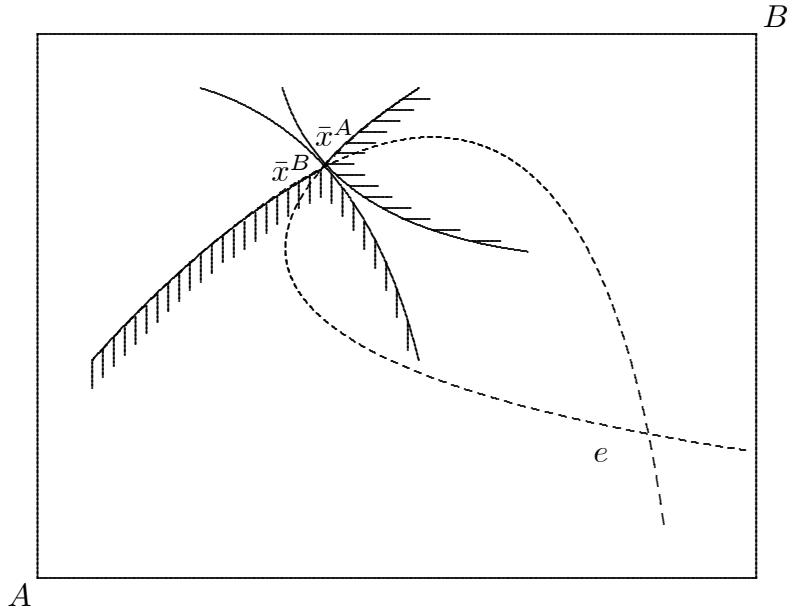
As a consequence of the previous lemmas, a SSP equilibrium with immediate acceptance can be characterized according to condition (4) by two allocations  $(x^A, e - x^A)$  and  $(e - x^B, x^B)$  (not necessarily distinct) on the agents' modified

offer curves (by Lemma 2) and such that agent  $A$  obtains from accepting the allocation  $(x^A, e - x^A)$  the same utility as he obtains from waiting one period for  $B$  to accept  $(e - x^B, x^B)$  and viceversa.

Consider in particular the case where  $\delta^A, \delta^B = 1$ . Since by Lemma 1,  $x^A$  and  $x^B$  have to be such that each agent leaves the other indifferent between accepting immediately and delaying the reach of an agreement, then  $x^A$  and  $x^B$  have to be on the same indifference curves for both agents. Moreover, by Lemma 2,  $x^A$  (resp.  $x^B$ ) must be either on  $A$ 's (resp.  $B$ 's) offer curve or on the contract curve above (resp. below) the Walrasian allocation. This allows us to establish (in the proof of Theorem 1 below) that every SSP equilibrium with immediate acceptance has to be Walrasian. In addition, it can easily be seen that  $x^A$  and  $x^B$  satisfy these conditions simultaneously only if they coincide with the Walrasian allocation. We turn to this first and then argue intuitively below that every SSP equilibrium has to be Walrasian.

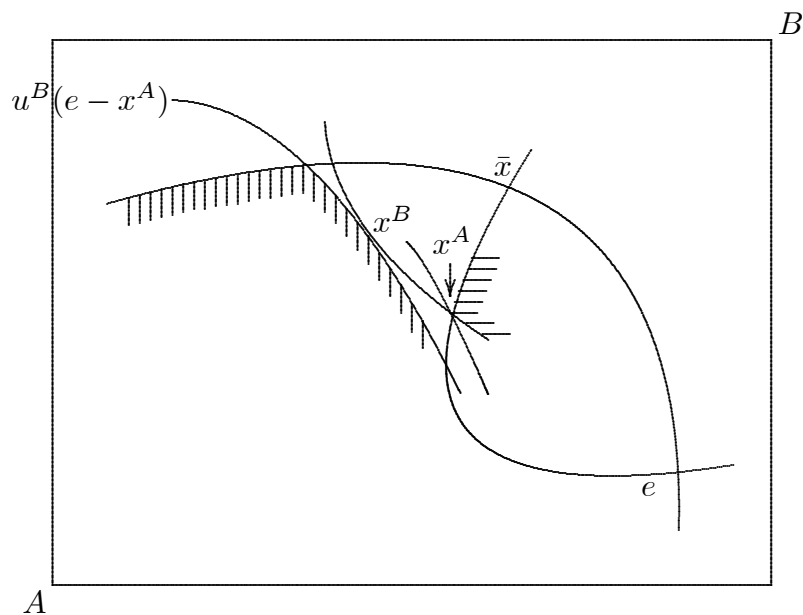
Firstly, for infinitely patient agents (i.e.  $\delta^A, \delta^B = 1$ ), the Walrasian allocations like  $(\bar{x}^A, \bar{x}^B)$  in Figure 4 below are the only allocations of SSP equilibria with immediate acceptance. Note that  $\bar{x}^A$  in Figure 4 maximizes indeed  $B$ 's utility within  $B$ 's choice set, which consists of the allocations "within"  $A$ 's offer curve guaranteeing  $A$  at least  $u^A(e - \bar{x}^B)$  (the horizontally shaded area) and, similarly,  $\bar{x}^B$  maximizes  $A$ 's utility within  $A$ 's choice set, i.e. the allocations "within"  $B$ 's offer curve guaranteeing  $B$  at least  $u^B(e - \bar{x}^A)$  (the vertically shaded area).

Figure 4



Secondly, for infinitely patient agents, there is no other allocation that corresponds to a SSP equilibrium with immediate acceptance. Assume on the contrary there is an allocation  $(x^A, x^B)$  corresponding to a SSP equilibrium with immediate acceptance. For instance, assume that  $B$  is able to guarantee himself a non-Walrasian level of utility  $u(e - x^A)$  as shown in Figure 5 below. Then  $x^B$  would necessarily have to be efficient, which leaves  $B$  a constrained set from which to choose her offer  $x^A$  within which she cannot actually reach the utility  $u(e - x^A)$ !

Figure 5



Theorem 1 summarizes the previous argument.

**Theorem 1.** *Every allocation of a SSP equilibrium with immediate acceptance is Walrasian, and conversely, whenever the agents are infinitely patient.*

The convergence towards Walrasian equilibrium allocations of the allocations of the SSP equilibria with immediate acceptance as the agents become arbitrarily patient is then the consequence of Theorem 1 and the upper hemicontinuity of the correspondence mapping the agents' discount factors to the set of allocations of the SSP equilibria with immediate acceptance. This result is stated as Theorem 2 below.

**Theorem 2.** *Every allocation of a SSP equilibrium with immediate acceptance converges to a Walrasian allocation as the agents become arbitrarily patient.*

#### 4. SSP EQUILIBRIA WITH DELAY?

So far, we have concentrated attention on SSP equilibria with immediate acceptance. For the remainder of this section, we will analyze SSP equilibria with delay in which one of the players rejects the offer made by the other player. It is well known that in generalized bargaining environments with complete information, equilibria with delay in reaching an agreement may exist (for an exhaustive treatment, see Merlo and Wilson (1996)). In the bargaining procedure considered here, it turns out that there is no SSP equilibrium with delay when the agents are impatient. In order to see this, we first extend the definition of a SSP equilibrium in (1) and (2) to allow for a delay in the acceptance of an offer. Without loss of generality, and in line with the argument above, we define equilibrium in terms of the allocations  $x^A, x^B$  that result from offers  $p^A, q^A$  and  $p^B, q^B$ .

In general, a SSP equilibrium consists of two allocations  $x^A$  and  $x^B$  (offered by  $B$  and  $A$  respectively) and acceptance rules according to which  $A$  rejects any offer

that does not allow him to attain at least a utility  $\delta^A u^A(e - x^B)$ , and similarly for  $B$ , that satisfy:

- (1)  $A$ 's offer  $x^B$  is rational, that is to say, given  $B$ 's offer  $x^A$ , either  $A$  prefers  $B$ 's acceptance of  $x^B$  to  $B$ 's rejection, i.e.  $u^A(e - x^B) > \delta^A u^A(x^A)$  and is making his most preferred offer acceptable to  $B$ , i.e.  $x^B$  solves

$$\begin{aligned} \max u^A(e - x^B) \\ Du^B(x^B)(x^B - e^B) \geq 0 \\ u^B(x^B) \geq \delta^B u^B(e - x^A) \end{aligned}$$

given  $x^A$ , or else  $A$  prefers  $B$ 's rejection of  $x^B$  to  $B$ 's acceptance, i.e.  $u^A(e - x^B) \leq \delta^A u^A(x^A)$  and is accordingly making an unacceptable offer to  $B$ ,

$$u^B(x^B) < \delta^B u^B(e - x^A)$$

and

- (2)  $B$ 's reply to  $A$ 's offer is rational, i.e.  $u^B(x^B) \geq \delta^B u^B(e - x^A)$  if  $B$  accepts  $x^B$ , and  $u^B(x^B) < \delta^B u^B(e - x^A)$  if  $B$  rejects  $x^B$ .

and similarly for  $B$ 's offer  $x^A$ .

Consider now a candidate SSP equilibrium in which say  $B$  rejects and  $A$  accepts. In such equilibrium,  $A$  prefers to make an offer  $x^B$  that will be rejected by  $B$ , and hence it gives  $A$  a smaller utility  $u^A(e - x^B)$  than the discounted utility  $\delta^A u^A(x^A)$  he derives from  $B$ 's offer  $x^A$ , i.e.

$$\delta^A u^A(x^A) \geq u^A(e - x^B).$$

Also  $A$  prefers to accept  $B$ 's offer  $x^A$  because it gives  $A$  a higher utility  $u^A(x^A)$  than his discounted utility  $\delta^A u^A(e - x^B)$  from his own offer, i.e.

$$u^A(x^A) \geq \delta^A u^A(e - x^B).$$

Then, for  $\delta^A < 1$ , this last inequality cannot be binding. That is to say, in a SSP equilibrium in which  $B$  rejects and  $A$  accepts,  $B$ 's offer  $x^A$  leaves  $A$  more utility than what would be necessary for having  $A$  accept it.

As a consequence, the second constraint of  $B$ 's problem is not binding, but since agent  $B$ 's preferences are monotone, then necessarily the first constraint constraint will be binding at  $x^A$ , i.e.

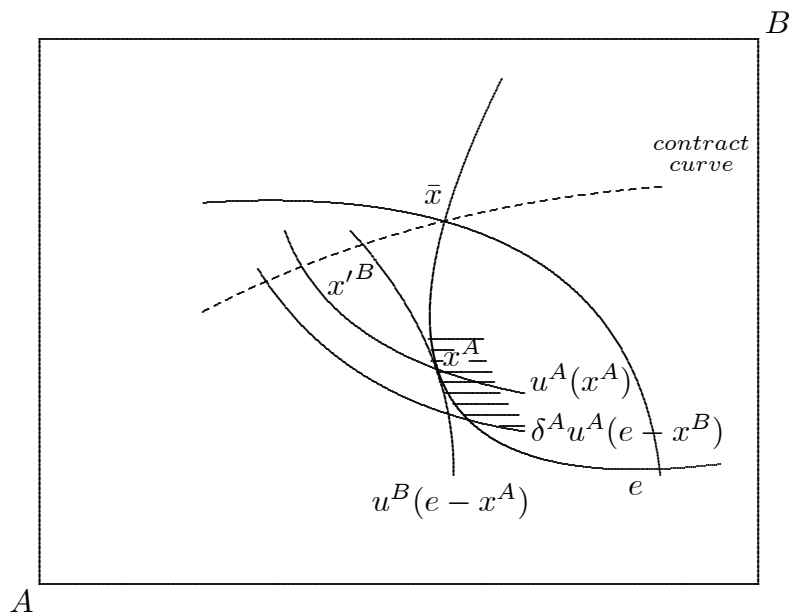
$$Du^A(x^A)(e^A - x^A) = 0$$

and therefore  $x^A$  has to be on  $A$ 's offer curve. Finally, this implies that there always exists a profitable deviation by  $A$  in this candidate equilibrium.

To see this, consider first the case where  $x^A$  is not efficient. Then there would be room for agent  $A$  to deviate profitably offering another  $x'^B$  which, if accepted by  $B$ , makes both  $A$  and  $B$  better off. This situation is illustrated in Figure 6 below. It is easy to see that any offer  $x'^B$  by  $A$  within the lens formed by the indifference

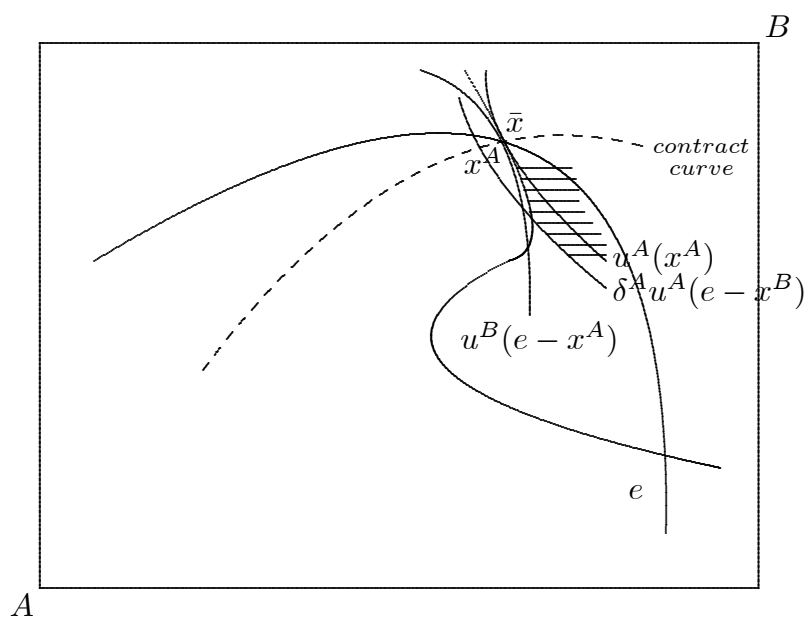
curves corresponding to  $u^A(x^A)$  and  $u^B(e - x^A)$  will lead  $B$  to accept the offer. That is to say, there is a mutually beneficial deviation of  $A$  from offering  $x^B$ .

Figure 6



Consider now the case in which  $x^A$  is efficient (and hence actually Walrasian) as in figure 7 below. Then, although there is no allocation that makes both agents better off, there is still an obvious profitable deviation by  $A$  that consists of preempting  $B$ 's offer by means of offering  $x^A$  himself from the beginning, since he will accept it anyway, after  $B$ 's rejection, paying unnecessarily the cost of the delay .

Figure 7



Therefore there cannot be a SSP equilibrium with delay for this bargaining game

if the agents exhibit the slightest degree of impatience.<sup>13</sup> This is stated in the next lemma.

**Lemma 3.** *Whenever the agents are impatient ( $\delta^A, \delta^B < 1$ ), there does not exist any SSP equilibrium with delay.*

In the absence of SSP equilibria with delay, we can now conclude that all SSP equilibria converge to the Walrasian allocation. The proof follows immediately from Lemma 3 and Theorem 2.

**Theorem 3.** *Every SSP equilibrium allocation converges to a Walrasian allocation as the agents become arbitrarily patient.*

## 5. DISCUSSION

In this section, we put our main result in the context of alternative bargaining procedures. As was mentioned in the introduction, Edgeworth (1882) points out that depending on the specifics of the bargaining procedure any Pareto improving allocation on the contract curve can be obtained. First, we illustrate how the standard alternating-offer bargaining outcome depends on the discount factors, even as they converge to one. We then point out that also in the case of Nash bargaining, the bargaining power of each agent determines the solution. We then proceed to argue that bargaining *over prices only* is not sufficient to obtain a bargaining foundation for the Walrasian equilibrium in small economies, and that a quantity constraint is needed. We also illustrate using an example that the quantity constraint must be a maximum trade, as other SSP equilibria exist when the constraint is a minimum trade constraint. And finally, we discuss the case of a fixed cost of delay as opposed to discounting.

**Bargaining over Allocations.** In the standard alternating-offer bargaining model (Rubinstein (1982) and Ståhl (1972)), proposers offer an allocation by means of a share of the surplus that is up for negotiation. As a result, the offer corresponds to a pair of utilities in which each agent consumes the share agreed upon. Rubinstein (1982) shows that, for discount factors strictly less than one, there exists a unique subgame perfect equilibrium of the bargaining game and that it is efficient.

Let  $x^A$  (resp.  $x^B$ ) denote a consumption offered by agent  $B$  to agent  $A$  (resp. by  $A$  to  $B$ ). An offer  $x^B$  made by agent  $A$  must be such that, in equilibrium, it leaves  $B$  with at least the same utility as what she could get in the next period:  $u^B(x^B) \geq \delta^B u^B(e - x^A)$ . Likewise for an offer by  $B$ :  $u^A(x^A) \geq \delta^A u^A(e - x^B)$ . Since the agents making an offer will always extract all the rents available to leave the receiver indifferent between accepting and rejecting, in equilibrium these inequalities hold with equality (see Rubinstein (1982)).<sup>14</sup> To represent these equilibrium

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<sup>13</sup>When  $\delta^A = \delta^B = 1$ , a continuum of SSP equilibria with delay exist (see Appendix). This is reminiscent of the continuum of equilibria in the Rubinstein alternating offer bargaining game without discounting and in which subgame perfect equilibria coincide with Nash equilibria.

<sup>14</sup>Note that in Lemma 1 above we have established a similar property of SSP equilibrium with immediate acceptance in the bargaining problem we analyze in this paper.

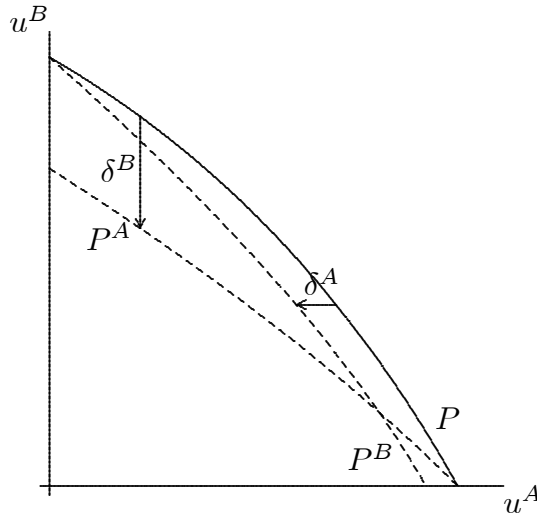
conditions, we can define therefore two curves of utility profiles  $P^A$  and  $P^B$  that are transformations of the Pareto frontier  $P$

$$P^A = (u^A(x^A), \delta^B u^B(x^B))$$

$$P^B = (\delta^A u^A(x^A), u^B(x^B)).$$

For example,  $P^A$  indicates the profile of utilities  $(u^A(x^A), \delta^B u^B(x^B))$  for all feasible allocations  $(x^A, x^B)$ , i.e. with  $x^A + x^B = e$ , and where  $B$ 's utility is discounted by  $\delta^B$  while  $A$ 's utility is unchanged. Likewise for  $P^B$ . An equilibrium corresponds therefore to an intersection of the curves of utility profiles  $P^A$  and  $P^B$ . This is illustrated in Figure 8.

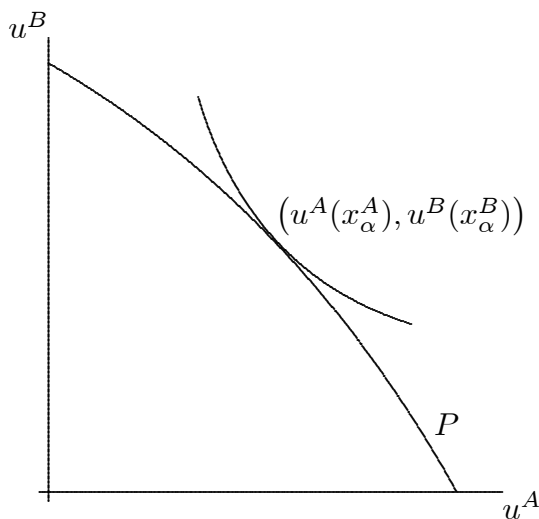
Figure 8



Clearly, the solution of the standard alternating offer bargaining game depends on the discount factors  $\delta^A, \delta^B$ . For example, as  $A$  becomes more and more patient ( $\delta^A$  goes to one),  $P^B$  converges to the Pareto frontier  $P$ . If  $\delta^B$  is kept constant, then the bargaining solution will be such that  $A$  extracts all the rents as the intersection of  $P^A$  and  $P^B$  converges to intersection of  $P$  with the horizontal axis. More generally, the outcome of the bargaining as the agents become infinitely patient depends then on the *rate* at which each of the  $\delta$ 's converge to one. If  $\delta^A$  converges faster to one, then the outcome will be proportionally more favorable to  $A$ .

**Nash Bargaining.** The same remark applies to the axiomatic bargaining solution proposed by Nash (1953). The Nash solution depends on the details of the bargaining procedure, both the outside option and the bargaining power of each of the agents. Here we will briefly illustrate the dependence on the relative bargaining power of each of the agents. Let  $\alpha$  and  $1 - \alpha$  be the bargaining power of agent  $A$  and  $B$  respectively. The Nash program then chooses, for a given  $\alpha \in (0, 1)$  the allocation  $(x_\alpha^A, x_\alpha^B)$  that maximizes the Nash product  $u^A(x^A)^\alpha \cdot u^B(x^B)^{1-\alpha}$  subject to the feasibility condition  $x^A + x^B = e$ . The solution is graphically illustrated in Figure 9 below.

Figure 9



As  $\alpha$  changes, the point on the Pareto frontier  $P$  that is selected as the solution to the bargaining problem changes. For  $\alpha$  converging to one for example, the solution converges to the intersection of  $P$  with the horizontal axis, i.e. agent  $A$  obtains the entire surplus. Binmore, Rubinstein and Wolinsky (1986) show that the alternating-offer bargaining solution approaches the Nash bargaining solution with utilities that reflect the incentives to settle and with the proper disagreement point chosen.

**Bargaining over prices only.** The only difference between the problems faced by the agents in the bargaining protocol in the current paper and the one we considered in Dávila and Eeckhout (2002) is precisely the presence of the quantity constraint. The agent making an offer here has the ability to limit the quantities to be traded chosen by the agent accepting it. This difference turns out to be crucial to get the convergence of the SSP equilibria of the bargaining game to a Walrasian outcome as the agents become infinitely patient. In order to see this, let us review the unconstrained case studied in Dávila and Eeckhout (2002) first, and then what changes in the constrained case.

As before, we can consider the problem of the agents as if they were bargaining over the allocations implied by the bargaining over the relative price, instead of bargaining over the price itself. More specifically, note that whenever agent  $A$  makes an offer of a relative price  $p^A$  he is effectively offering the allocation  $(e - x^B(p^A), x^B(p^A))$ , where  $x^B(p^A)$  is the usual (unconstrained) demand of  $B$  at the relative price  $p^A$ . In other words, in the unconstrained demand case  $A$  is actually constrained to offer allocations on  $B$ 's offer curve, i.e. an  $x^B$  such that

$$Du^B(x^B)(x^B - e^B) = 0.$$

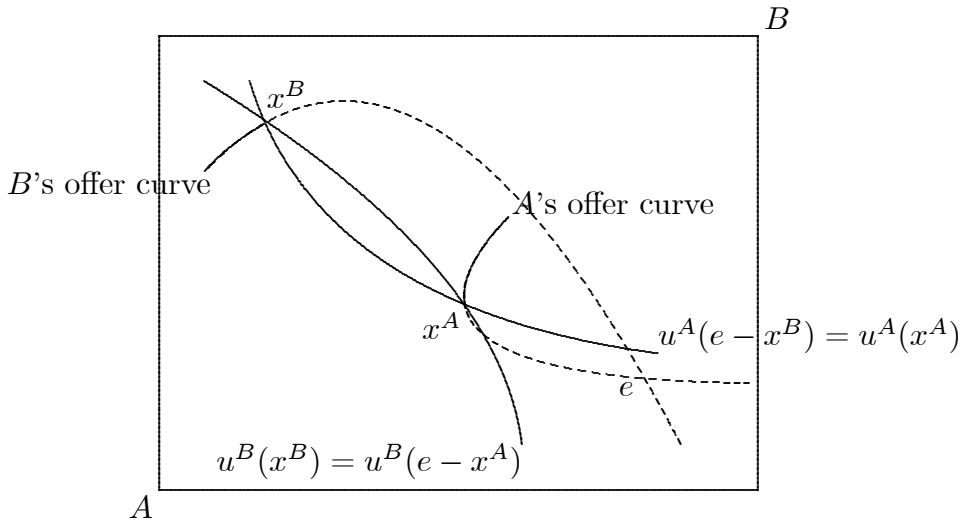
Now there is no longer any quantity constraint. Therefore an SSP equilibrium of the bargaining over prices only can also be characterized as a consumption  $x^A$  offered by  $B$  to  $A$  and similarly a consumption  $x^B$  offered by  $A$  to  $B$  such that  $x^A$  solves

$$\begin{aligned} \max u^B(e - x^A) \\ Du^A(x^A)(x^A - e^A) = 0 \\ u^A(x^A) \geq \delta^A u^A(e - x^B) \end{aligned}$$

given  $x^B$ , and likewise for  $B$ .

This implies that any solution must be on each of the the accepting agent's offer curve. When agents are infinitely patient, i.e. for  $\delta^A, \delta^B = 1$ , the utility that any of the agents receives must be the same whether he accepts an offer or whether his offer is accepted. It is straightforward that the Walrasian allocation  $(\bar{x}^A, \bar{x}^B)$  solves the problems above for infinitely patient agents. What is not that obvious is that, generically in the space of preferences,<sup>15</sup> there are inefficient allocations  $(x^A, e - x^A)$  and  $(e - x^B, x^B)$  that satisfy the conditions for a stationary subgame perfect equilibrium with immediate acceptance as well. The existence of such allocations is proved in Dávila and Eeckhout (2002),<sup>16</sup> and the following Figure 10 exhibits one such pair of allocations.

Figure 10



Note that in Figure 10, in effect,  $B$ 's offer  $x^A$  maximizes  $B$ 's utility over the piece (in solid) of  $A$ 's offer curve (in dashes) corresponding to levels of utility for  $A$  at least as big as  $u^A(e - x^B)$ , and  $A$ 's offer  $x^B$  maximizes  $A$ 's utility over the piece (in solid) of  $B$ 's offer curve (in dashes) corresponding to levels of utility for  $B$  at least as big as  $u^B(e - x^A)$ .

By continuity, for discount factors smaller than but close enough to 1, there is a stationary subgame perfect equilibrium whose allocations (depending on who accepts whose offer) are close to the inefficient allocations  $(x^A, e - x^A)$  and  $(e - x^B, x^B)$ , and moreover converge to these inefficient allocations as the agents become arbitrarily patient. Hence the lack of convergence towards a Walrasian outcome of the outcome of bargaining only over prices.

To further illustrate this lack of convergence to the Walrasian equilibrium allocation if the agents bargain over prices only, we represent the utility pairs obtained along the offer curves in Figure 11. The "inverted-U-shaped" solid curve labelled

<sup>15</sup>More precisely, for an open and dense subset of utility functions with respect to the adequate topology.

<sup>16</sup>The conditions above characterizing these allocations are in principle only necessary. We studied in Dávila and Eeckhout (2002) when they are sufficient a well.

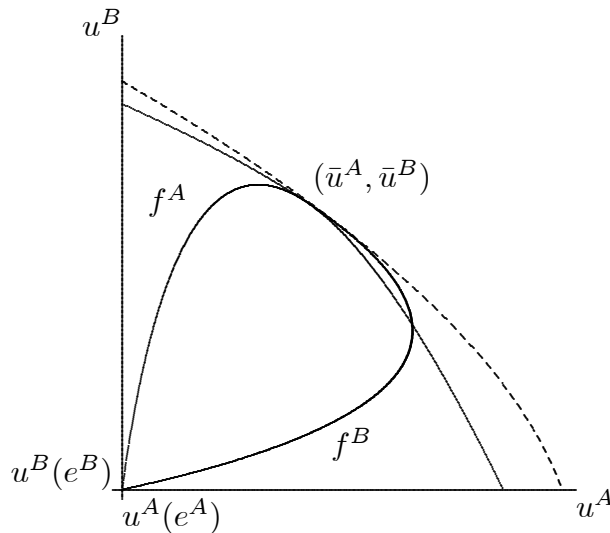
$f^A$  represents the utility combinations of both agents as we run along  $A$ 's offer curve. Initially, both agents' utility increases, to reach a maximum at the monopoly allocation. Thereafter,  $B$ 's utility decreases as  $A$ 's utility continues to increase. Similarly the "left-U-shaped" solid curve  $f^B$  represents the pairs of utility running along  $B$ 's offer curve. The curve in dashes represents the Pareto frontier.

An intersection of both curves  $f^A$  and  $f^B$  corresponds<sup>17</sup> to a SSP equilibrium when bargaining only over prices in the case where  $\delta^A, \delta^B = 1$ , since it would amount to the satisfaction of the equations

$$\begin{aligned} u^A(x^A) &= u^A(e - x^B) \\ u^B(e - x^A) &= u^B(x^B) \end{aligned}$$

by an  $x^A$  on  $A$ 's offer curve and an  $x^B$  on  $B$ 's offer curve. The Walrasian allocation is hence a SSP equilibrium in this case also, but there is another intersection that is strictly inside the Pareto frontier. In Dávila and Eeckhout (2002) we show that this inefficient SSP equilibrium robustly exists even for  $\delta^A, \delta^B < 1$ . On the contrary, as we also show there, with discounting the SSP equilibrium allocation converging to the Walrasian allocation is no longer guaranteed to exist uniquely, or even to exist at all: either there is no such SSP equilibrium (the intersection at  $(\bar{u}^A, \bar{u}^B)$ ) vanishes as  $\delta^A, \delta^B$  depart from 1), or if it exists, there is an even number of them (the intersection at  $(\bar{u}^A, \bar{u}^B)$ ) bifurcates as  $\delta^A, \delta^B$  depart from 1). Which case arises depends on the path followed by the discount factors towards one.

Figure 11



Note that in the case of bargaining over prices with quantity constraints this inefficient allocation ceases to be an equilibrium. To see this, observe that (i) offers immediately accepted by  $A$  at a SSP equilibrium that provide  $A$  a utility that is lower than his Walrasian utility  $\bar{u}^A$  will not be bound by the quantity constraint and, as was shown above (see Lemma 2), they will be on  $A$ 's offer curve, while (ii) offers made to  $A$  that provide  $A$  a utility that is higher than  $\bar{u}^A$  will be bound by

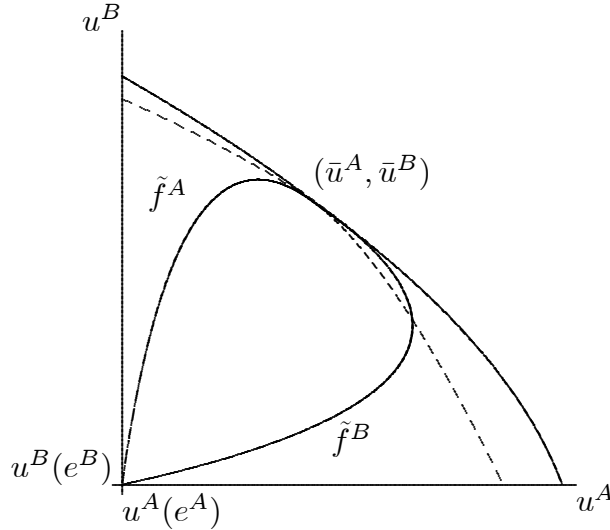
<sup>17</sup>Provided both curves have negative slopes, which guarantees the subgame perfection of the equilibrium.

the quantity constraint, and hence they will be on the contract curve. Likewise for offers accepted by  $B$ .

Therefore, a SSP equilibrium of bargaining over prices with quantity constraints corresponds to the intersection of the two curves  $\tilde{f}^A$  and  $\tilde{f}^B$  shown in Figure 12 below. These curves are composed partly of the offer curve ( $f^A$  to the left of  $(\bar{u}^A, \bar{u}^B)$  for offers accepted by  $A$ ) and partly of the Pareto frontier (from  $(\bar{u}^A, \bar{u}^B)$  to its right, for offers accepted by  $A$ , and similarly for  $B$ ). In effect, in the case where  $\delta^A, \delta^B = 1$ , such an intersection amounts to the following equations being satisfied by a point  $\tilde{x}^A$  on  $A$ 's *modified* offer curve and a point  $\tilde{x}^B$  on  $B$ 's *modified* offer curve.

$$\begin{aligned} u^A(\tilde{x}^A) &= u^A(e - \tilde{x}^B) \\ u^B(e - \tilde{x}^A) &= u^B(\tilde{x}^B) \end{aligned}$$

Figure 12



It follows straightforwardly then that only the Walrasian outcomes correspond to SSP equilibria of the bargaining game when  $\delta^A, \delta^B = 1$ . Note also that every Walrasian intersection is robust now with the bargaining over prices *and quantity constraints* to slight departures of  $\delta^A, \delta^B$  from 1. The previous discussion also shows that the convergence result does not depend on having just two goods, since independently of the number of goods, in the space of utilities the same pattern would appear.

**Bargaining over prices with minimum quantity constraints.** Is there anything special about the bargaining procedure of alternating price offers with maximum quantity constraints? As discussed earlier, the quantity constraint is crucial to guarantee convergence to the Walrasian allocation. But can a case be made for using *minimum* quantity constraints, instead of maximum constraints? The only difference is that whenever an agent, say  $A$ , is interested in having his offer accepted at equilibrium, this offer  $x^B$  has to be the solution to the only slightly

different problem below

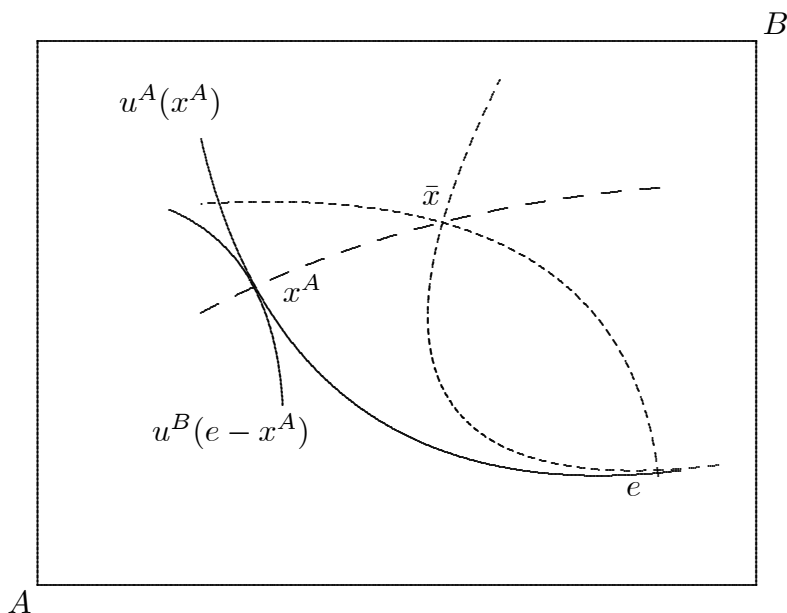
$$\begin{aligned} \max u^A(e - x^B) \\ Du^B(x^B)(x^B - e^B) \leq 0 \\ u^B(x^B) \geq \delta^B u^B(e - x^A) \end{aligned}$$

given  $x^B$ . Observe that compared to the problem solved by the agent proposing a maximum quantity constraint, the inequality in the first constraint is now reversed. This means that agent  $B$  may be prevented from choosing as little as he may desire to trade at some offered price. The proofs of Theorems 1 and 2 can be straightforwardly modified to show that they still hold with minimum rather than maximum quantity constraints. As a result, there is a SSP equilibrium with immediate acceptance converging to the Walrasian allocation.

The main difference however is that this is no longer the unique SSP equilibrium because there also exist SSP equilibria with delay when agents are not perfectly patient. To see this, consider the same candidate SSP equilibrium as above in which  $B$  rejects the offer  $x^B$  made by  $A$ , and  $A$  accepts the offer  $x^A$  made by  $B$ . For the case  $\delta^A, \delta^B = 1$ , we show that the  $x^A$  on the contract curve such that  $u^A(x^A) = u^A(e^A)$  is indeed an SSP equilibrium with delay. This is illustrated in Figure 13 below.

Note that, independently of the unacceptable offer  $x^B$  that  $A$  may make to  $B$ , by means of imposing a minimum trade, agent  $B$  can make an offer that forces  $A$  to accept on the contract curve even though  $A$  would prefer to accept on his offer curve. This implies that the offer made by  $B$  and accepted by  $A$  is efficient. As a result, there does not exist any counter-offer  $x^B$  by  $A$  that  $B$  will accept and that makes both better off. Therefore, there is no profitable deviation. By continuity, the same is true for discount factors close enough to 1.

Figure 13



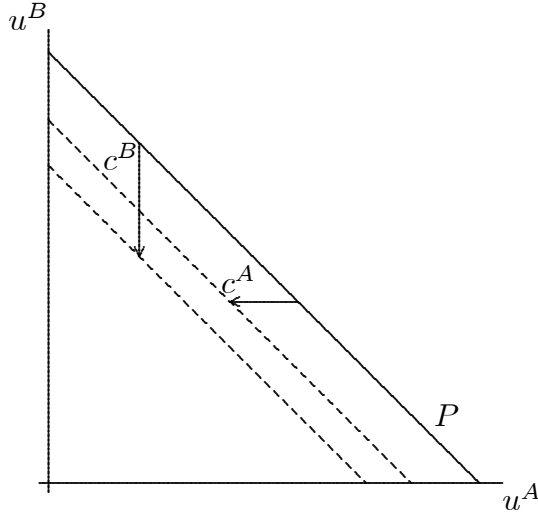
Clearly, the same argument can be applied to establish that there will exist a SSP equilibrium with delay where  $A$ 's offer  $x^B$  is accepted by  $B$  and  $B$ 's offer is rejected

by  $A$ . The offer  $x^B$  is on the contract curve and it satisfies  $u^B(x^B) = u^B(e^B)$ . This is consistent with the characterization of SSP equilibria with delay in Merlo and Wilson (1995).

**Bargaining with a constant cost of delay.** In many environments bargaining parties have preferences for early settlement, but the cost of delay is constant. This is in contrast with discounting, where the cost of delay is proportional to the value of the surplus. Examples with a fixed transaction cost per unit of time in the process of bargaining include bargaining over the phone with a per unit of time cost of the call, or bargaining that is executed through lawyers who bill by the hour. Therefore, preferences may be more accurately represented by  $u^A(z^A) - tc^A$ , where  $z^A$  is a feasible allocation,  $c^A$  is the utility cost of delaying one period, and  $t$  is the number of periods gone by. Similarly for  $B$ .

For the case of the standard Rubinstein bargaining model with a pie of size 1 and linear preferences  $z^A - tc^A$ , fixed costs of delay produce a radically different result than discounting (see Rubinstein (1982)). Two necessary conditions for a subgame perfect equilibrium are  $z^B = 1 - z^A - c^B$  and  $z^A = 1 - z^B - c^A$ , where  $z^A$  is the offer accepted by  $A$  when  $B$  makes the offer, and  $c^B$  is the cost to  $B$  of delaying one period. Basically, if  $B$  accepts  $A$ 's offer, she has to be indifferent between  $z^B$  and  $1 - z^A - c^B$ . Let  $c^A < c^B$ , then it is easy to see in Figure 14 below that there is no intersection of the horizontally and vertically translated Pareto frontier. Rubinstein (1982) shows that in that case, player  $A$  who is less impatient will get the entire surplus. This remains true as the costs of delay vanish, whatever the rate at which  $c^A$  and  $c^B$  converge to 0.

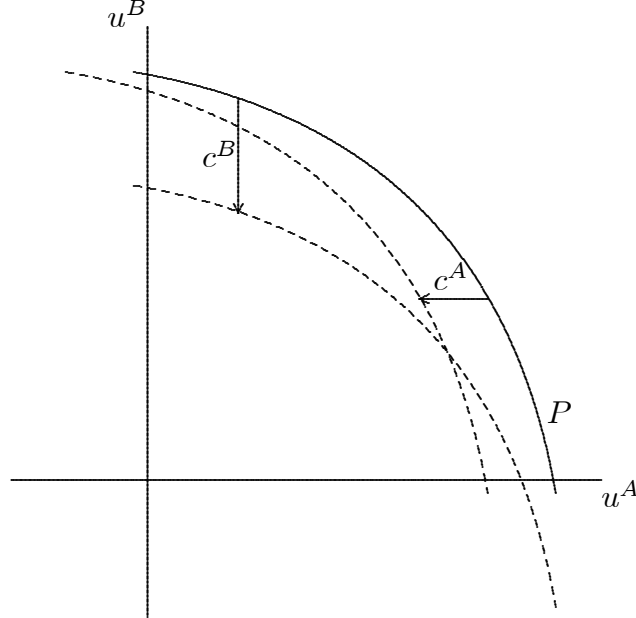
Figure 14



The fact that with constant costs the most patient agent gets the entire surplus (as in Rubinstein (1982)) is quite specific to the case of linear preferences. This no longer holds for more general preferences. Consider for example the case where the utility is not linear, but rather the natural logarithm plus a positive constant (the constant, when large enough, ensures positive utilities) for both agents. Then necessary conditions are  $\ln(x^B) = \ln(1 - x^A) - c^B$  and  $\ln(x^A) = \ln(1 - x^B) - c^A$  and imply  $x^B = e^{-c^B} (1 - x^A)$  and  $x^A = e^{-c^A} (1 - x^B)$ . Now for  $c > 0$ , we have that

$e^{-c} < 1$ . Therefore, let  $d^A = e^{-c^A}$  (similarly for  $B$ ) and we get exactly the same solution as in the case the standard Rubinstein bargaining game with discounting, where  $d^A > d^B$ . This is illustrated in Figure 15:

Figure 15



As  $d^A$  and  $d^B$  converge to 1 (i.e.  $c^A, c^B$  converge to 0), the solutions will be interior and equal to  $((1/(1+r)), (r/(1+r)))$  where  $r$  is the rate at which  $d$ 's converge to 1:  $r = \ln d^A / \ln d^B = c^A / c^B$ . This solution is always interior for  $c^A, c^B > 0$ . What this example illustrates is that the "corner" solution in which the more patient agent gets the entire surplus when there is a constant cost of delay is not generic.<sup>18</sup>

The same is true in our model of bargaining over prices with maximum quantity constraints and with general preferences when fixed costs of delay are introduced instead of discounting. The problem in (4) now can be written as

$$\begin{aligned} \max u^A(e - x^B) \\ Du^B(x^B)(x^B - e^B) &\geq 0 \\ u^B(x^B) &\geq u^B(e - x^A) - c^B. \end{aligned} \quad (4')$$

Obviously, problem (4') without cost of delay ( $c = 0$ ) and problem (4) without discounting frictions ( $\delta = 1$ ) are identical. Therefore Theorem 1 immediately extends to the case  $c^A = c^B = 0$ . We do not provide a proof for the counterpart of Theorem 2, but observe that the proof of Theorem 2 is built on a continuity argument of the equilibrium correspondence as discounting frictions vanish. Graphically, in the space of utilities, Theorem 1 derives from the intersections of two graphs. With small discounting frictions, the SSP equilibria with immediate acceptance correspond to the intersection of these slightly transformed graphs. Whether that

<sup>18</sup>For a discussion on more general classes of intertemporal preferences and the axiomatic approach to delay, see Osborne and Rubinstein (1990).

transformation is proportional (as in the case of discounting) or additive (as in the case of fixed costs) does not alter the outcome.

## 6. CONCLUDING REMARKS

In this paper, we have proposed a simple bargaining procedure that achieves the competitive equilibrium allocation without assuming price-taking behavior. The procedure is commonly observed, in the sense that negotiating parties often bargain over a price, and that the quantity of trade is chosen separately. The main contribution of this paper is to show that by always obtaining the Walrasian equilibrium, the outcome of the bargaining does not depend on specifics such as relative bargaining powers or impatience, but only on primitives, i.e. preferences and endowments.

This bargaining procedure is of particular appeal when applied in economic theories in which bargaining is a natural component of pairwise negotiation. In search theory for example, the effectiveness of a policy that restores efficiency (see Hosios (1990)) no longer depends on some exogenous parameter that is often hard to observe, let alone quantify. Rather, the determinacy of the bargaining outcome implies that an efficient policy is a function only of the preferences and endowments of the agents in the economy.

The results derived here hold for economies with two agents only, which makes the price-taking behavior assumption hard to justify. It remains an open question whether this bargaining procedure will still generate the Walrasian outcome when the number of agents is larger than two. We do know though that for the standard alternating-offer bargaining game à la Rubinstein, the equilibrium in stationary strategies is unique for games with more than two agents.<sup>19</sup> Even so, it remains to be verified whether our bargaining procedure for more than two agents will lead to the competitive outcome. At the other extreme, when the number of agents increases to a continuum and pairs are formed through random matching, then the bargaining procedure proposed in Gale (1986) will lead to the Walrasian equilibrium outcome.

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<sup>19</sup>See Herrero (1985) and Sutton (1986). However, in non-stationary strategies, uniqueness is not guaranteed.

APPENDIX

**Lemma 1.** *At every SSP equilibrium with immediate acceptance  $p^A, q^A, p^B, q^B$ ,  $A$  leaves  $B$  not more utility than is strictly necessary to obtain his acceptance, that is to say*

$$u^B(\tilde{x}^B(p^A, q^A)) = \delta^B u^B(e - \tilde{x}^A(p^B, q^B))$$

and hence, from  $A$ 's problem's first order conditions

$$[Du^A(e - \tilde{x}^B(p^A, q^A)) - \lambda^A Du^B(\tilde{x}^B(p^A, q^A))] D\tilde{x}^B(p^A, q^A) = 0$$

for some  $\lambda^A \geq 0$ . Likewise for  $B$ .

*Proof.* Let  $(p^A, q^A, p^B, q^B)$  be an SSP equilibrium with immediate acceptance, then necessarily

$$\begin{aligned} (p^A, q^A) &\in \arg \max u^A(e - \tilde{x}^B(\hat{p}^A, \hat{q}^A)) \\ u^B(\tilde{x}^B(\hat{p}^A, \hat{q}^A)) &\geq \delta^B u^B(e - \tilde{x}^A(p^B, q^B)) \end{aligned}$$

Since the constraint qualification condition is satisfied by the problem, the necessarily, for some  $\lambda^A \geq 0$ ,

$$\begin{aligned} [Du^A(e - \tilde{x}^B(p^A, q^A)) - \lambda^A Du^B(\tilde{x}^B(p^A, q^A))] D\tilde{x}^B(p^A, q^A) &\geq 0 \\ [Du^A(e - \tilde{x}^B(p^A, q^A)) - \lambda^A Du^B(\tilde{x}^B(p^A, q^A))] D\tilde{x}^B(p^A, q^A)(p^A, q^A) &= 0 \\ \lambda^A [u^B(\tilde{x}^B(p^A, q^A)) - \delta^B u^B(e - \tilde{x}^A(p^B, q^B))] &= 0 \end{aligned} \quad (1)$$

Note that

- (1) either the quantity constraint is binding for  $B$ , i.e.

$$\tilde{x}^B(p^A, q^A) \neq x^B(p^A)$$

- (2) or it is not, i.e.

$$\tilde{x}^B(p^A, q^A) = x^B(p^A)$$

and

- (3) either the solution to  $B$ 's problem is interior, i.e.

$$u^B(\tilde{x}^B(p^A, q^A)) > \delta^B u^B(e - \tilde{x}^A(p^B, q^B))$$

and

$$Du^A(e - \tilde{x}^B(p^A, q^A)) D\tilde{x}^B(p^A, q^A) = 0$$

- (4) or it is not, i.e.

$$u^B(\tilde{x}^B(p^A, q^A)) = \delta^B u^B(e - \tilde{x}^A(p^B, q^B))$$

and

$$[Du^A(e - \tilde{x}^B(p^A, q^A)) - \lambda^A Du^B(\tilde{x}^B(p^A, q^A))] D\tilde{x}^B(p^A, q^A) = 0$$

for some  $\lambda^A \geq 0$ .

Let us see that (3) cannot hold.

Assume first that (1) holds, i.e.  $\tilde{x}^B(p^A, q^A) \neq x^B(p^A)$ . Can (3) hold with (1)? No, it cannot. In effect, since  $\tilde{x}^B(p^A, q^A) \neq x^B(p^A)$ , then  $D_2\tilde{x}^B(p^A, q^A) \neq 0$  since the preferences are monotone. If moreover (3) was the case, since the gradient  $Du^A(e - \tilde{x}^B(p^A, q^A))$  is in  $\mathbb{R}_{++}^2$ , then

- (31) either both  $D_1\tilde{x}^B(p^A, q^A)$  and  $D_2\tilde{x}^B(p^A, q^A)$  are null,
- (32) or one of  $D_1\tilde{x}^B(p^A, q^A)$  and  $D_2\tilde{x}^B(p^A, q^A)$  is null and the other is non-null and orthogonal to  $Du^A(e - \tilde{x}^B(p^A, q^A))$ ,
- (33) or both  $D_1\tilde{x}^B(p^A, q^A)$  and  $D_2\tilde{x}^B(p^A, q^A)$  are non-null and orthogonal to  $Du^A(e - \tilde{x}^B(p^A, q^A))$ .

Cases (31) and (32) above can be discarded since  $D_2\tilde{x}^B(p^A, q^A) \neq 0$  and also  $D_1\tilde{x}^B(p^A, q^A) \neq 0$  for monotone preferences. As a consequence, (3) could only hold with (1) if both  $D_1\tilde{x}^B(p^A, q^A)$  and  $D_2\tilde{x}^B(p^A, q^A)$  are non-null and orthogonal to  $Du^A(e - \tilde{x}^B(p^A, q^A))$ , but this requires that  $D_1\tilde{x}^B(p^A, q^A)$  and  $D_2\tilde{x}^B(p^A, q^A)$  be collinear, which cannot be since  $D_1\tilde{x}^B(p^A, q^A)$  is in  $\mathbb{R}_+^2 \setminus \{0\}$  or  $\mathbb{R}_-^2 \setminus \{0\}$  and  $D_2\tilde{x}^B(p^A, q^A)$  is not (note that  $D_2\tilde{x}^B(p^A, q^A)$  is normal to  $(p^A, 1)$ , which is in  $\mathbb{R}_{++}^2$ ).

Assume now that (2) holds, i.e.  $\tilde{x}^B(p^A, q^A) = x^B(p^A)$ . Can (3) hold with (2)? Again no, it cannot. In effect, since  $\tilde{x}^B(p^A, q^A) = x^B(p^A)$ , then  $D_2\tilde{x}^B(p^A, q^A) = 0$ . If moreover (3) was the case, since  $Du^A(e - \tilde{x}^B(p^A, q^A)) \in \mathbb{R}_{++}^2$ , then again

- (31) either both  $D_1\tilde{x}^B(p^A, q^A)$  and  $D_2\tilde{x}^B(p^A, q^A)$  are null,
- (32) or one of  $D_1\tilde{x}^B(p^A, q^A)$  and  $D_2\tilde{x}^B(p^A, q^A)$  is null and the other is non-null and orthogonal to  $Du^A(e - \tilde{x}^B(p^A, q^A))$ ,
- (33) or both  $D_1\tilde{x}^B(p^A, q^A)$  and  $D_2\tilde{x}^B(p^A, q^A)$  are non-null and orthogonal to  $Du^A(e - \tilde{x}^B(p^A, q^A))$ .

Case (31) above can be discarded since  $D_1\tilde{x}^B(p^A, q^A) \neq 0$  for monotone preferences, and also (33) can be discarded since  $D_2\tilde{x}^B(p^A, q^A) = 0$  because (2) holds. As a consequence, (3) could only hold with (2) if  $D_1\tilde{x}^B(p^A, q^A)$  was orthogonal to  $Du^A(e - \tilde{x}^B(p^A, q^A))$ , but  $Du^A(e - \tilde{x}^B(p^A, q^A)) \in \mathbb{R}_{++}^2$  while  $D_1\tilde{x}^B(p^A, q^A)$  is in  $\mathbb{R}_+^2 \setminus \{0\}$  or  $\mathbb{R}_-^2 \setminus \{0\}$  and hence they cannot be orthogonal.

Since (3) cannot hold in any case, then necessarily (4) holds. Q.E.D.

**Lemma 2.** *For every SSP equilibrium with immediate acceptance  $p^A, q^A, p^B, q^B$ , if the offer accepted by A is not on his offer curve, then it is efficient. Likewise for B.*

*Proof.* We need to verify that, for every SSP equilibrium with immediate acceptance  $(p^A, q^A, p^B, q^B)$ , whenever  $\tilde{x}^A(p^B, q^B) \neq x^A(p^B)$ , the allocation  $(\tilde{x}^A(p^B, q^B), e - \tilde{x}^A(p^B, q^B))$  is efficient.

Let  $(p^A, q^A, p^B, q^B)$  be a SSP equilibrium with immediate acceptance. Then necessarily

$$\begin{aligned} (p^B, q^B) &\in \arg \max u^B(e - \tilde{x}^A(\hat{p}^B, \hat{q}^B)) \\ u^A(\tilde{x}^A(\hat{p}^B, \hat{q}^B)) &\geq \delta^A u^A(e - \tilde{x}^B(p^A, q^A)) \end{aligned}$$

Since the constraint qualification condition is satisfied by the problem, then necessarily, for some  $\lambda^B \geq 0$ ,

$$\begin{aligned} & [Du^B(e - \tilde{x}^A(p^B, q^B)) - \lambda^B Du^A(\tilde{x}^B(p^A, q^A))] D\tilde{x}^A(p^B, q^B) \geq 0 \\ & [Du^B(e - \tilde{x}^A(p^B, q^B)) - \lambda^B Du^A(\tilde{x}^B(p^A, q^A))] D\tilde{x}^A(p^B, q^B)(p^B, q^B) = 0 \quad (1) \\ & \lambda^B [u^A(\tilde{x}^A(\hat{p}^B, \hat{q}^B)) - \delta^A u^A(e - \tilde{x}^B(p^A, q^A))] = 0 \end{aligned}$$

Assume that  $\tilde{x}^A(p^B, q^B) \neq x^A(p^B)$  and that  $(\tilde{x}^A(p^B, q^B), e - \tilde{x}^A(p^B, q^B))$  is inefficient. Then, by the monotonicity of the preferences,  $D_2\tilde{x}^A(p^B, q^B) \neq 0$ . Moreover,  $D_2\tilde{x}^A(p^B, q^B)$  and  $D_1\tilde{x}^A(p^B, q^B)$  are linearly independent when  $D_2\tilde{x}^A(p^B, q^B) \neq 0$ , since  $D_1\tilde{x}^A(p^B, q^B)$  is in  $\mathbb{R}_-^2$  or  $\mathbb{R}_+^2$  while  $D_2\tilde{x}^A(p^B, q^B)$  is in none of them.

Since  $p^B, q^B > 0$ , then according to the first order conditions (1) there exists a linear combination  $Du^B(e - \tilde{x}^A(p^B, q^B)) - \lambda^B Du^A(\tilde{x}^B(p^A, q^A))$  of the gradients of utilities that

- (1) is orthogonal to a vector in the cone of positive linear combinations of  $D_1\tilde{x}^A(p^B, q^B)$  and  $D_2\tilde{x}^A(p^B, q^B)$ , and
- (2) has a non-negative inner product with  $D_1\tilde{x}^A(p^B, q^B)$  and  $D_2\tilde{x}^A(p^B, q^B)$ .

But these two conditions cannot be satisfied simultaneously. In effect, since the allocation  $(\tilde{x}^A(p^B, q^B), e - \tilde{x}^A(p^B, q^B))$  is inefficient, then the condition

$$[Du^B(e - \tilde{x}^A(p^B, q^B)) - \lambda^B Du^A(\tilde{x}^B(p^A, q^A))] D\tilde{x}^A(p^B, q^B)(p^B, q^B) = 0$$

can only hold with

$$[Du^B(e - \tilde{x}^A(p^B, q^B)) - \lambda^B Du^A(\tilde{x}^B(p^A, q^A))] D\tilde{x}^A(p^B, q^B) \geq 0$$

and  $(p^B, q^B) \gg 0$ , if

$$[Du^B(e - \tilde{x}^A(p^B, q^B)) - \lambda^B Du^A(\tilde{x}^B(p^A, q^A))] D\tilde{x}^A(p^B, q^B) = 0$$

but then  $D_2\tilde{x}^A(p^B, q^B)$  and  $D_1\tilde{x}^A(p^B, q^B)$  are linearly dependent!

Therefore, if  $\tilde{x}^A(p^B, q^B) \neq x^A(p^B)$ , then  $(\tilde{x}^A(p^B, q^B), e - \tilde{x}^A(p^B, q^B))$  is efficient. Q.E.D.

**Theorem 1.** *Every allocation of a SSP equilibrium with immediate acceptance is Walrasian, and conversely, whenever the agents are infinitely patient.*

*Proof.* Let  $(p^A, q^A, p^B, q^B)$  be a SSP equilibrium with immediate acceptance when the agents are infinitely patient, i.e. with  $\delta^A = 1 = \delta^B$ . Then

- (1) either for some agent, say  $A$ , his demand is effectively constrained, i.e.

$$\tilde{x}^A(p^B, q^B) \neq x^A(p^B)$$

- (2) or for both agents the demand is unconstrained, i.e.

$$\tilde{x}^A(p^B, q^B) = x^A(p^B)$$

$$\tilde{x}^B(p^A, q^A) = x^B(p^A).$$

If (1), then by Lemma 2 the allocation  $(\tilde{x}^A(p^B, q^B), e - \tilde{x}^A(p^B, q^B))$  is efficient and, since  $\tilde{x}^A(p^B, q^B) \neq x^A(p^B)$ , non-walrasian. Moreover,

$$(\tilde{x}^A(p^B, q^B), e - \tilde{x}^A(p^B, q^B)) = (e - \tilde{x}^B(p^A, q^A), \tilde{x}^B(p^A, q^A))$$

since by Lemma 1 the constraints are binding, i.e.

$$\begin{aligned} u^A(\tilde{x}^A(p^B, q^B)) &= u^A(e - \tilde{x}^B(p^A, q^A)) \\ u^B(e - \tilde{x}^A(p^B, q^B)) &= u^B(\tilde{x}^B(p^A, q^A)) \end{aligned}$$

which means that the allocation  $(\tilde{x}^A(p^B, q^B), e - \tilde{x}^A(p^B, q^B))$  and the allocation  $(e - \tilde{x}^B(p^A, q^A), \tilde{x}^B(p^A, q^A))$  are, within the Edgeworth box, located on the same indifference curve for each of the two agents. Since  $(\tilde{x}^A(p^B, q^B), e - \tilde{x}^A(p^B, q^B))$  is efficient, it is the only allocation providing  $A$  and  $B$  the levels of utility  $u^A(\tilde{x}^A(p^B, q^B))$  and  $u^B(e - \tilde{x}^A(p^B, q^B))$  respectively, from which the equality of the two allocations offered follows. But at an efficient, non-Walrasian allocation such that  $\tilde{x}^A(p^B, q^B) \neq x^A(p^B)$ , it holds

$$Du^B(e - \tilde{x}^A(p^B, q^B))(e^A - \tilde{x}^A(p^B, q^B)) < 0$$

because

$$Du^A(\tilde{x}^A(p^B, q^B))(\tilde{x}^A(p^B, q^B) - e^A) > 0$$

since  $\tilde{x}^A(p^B, q^B) \neq x^A(p^B)$ , and the gradients  $Du^A(\tilde{x}^A(p^B, q^B))$  and  $Du^B(\tilde{x}^B(p^A, q^A))$  are collinear because of the efficiency of the allocation. Nevertheless, it must also hold

$$Du^B(\tilde{x}^B(p^A, q^A))(\tilde{x}^B(p^A, q^A) - e^B) \geq 0$$

because of the maximum trade constraint in  $B$ 's choice upon acceptance of  $A$ 's offer, and

$$\tilde{x}^B(p^A, q^A) = e - \tilde{x}^A(p^B, q^B),$$

that is to say,

$$Du^B(e - \tilde{x}^A(p^B, q^B))(e^A - \tilde{x}^A(p^B, q^B)) \geq 0$$

which contradicts the inequality above. Therefore (1) cannot hold and necessarily (2) holds, i.e.

$$\begin{aligned} \tilde{x}^A(p^B, q^B) &= x^A(p^B) \\ \tilde{x}^B(p^A, q^A) &= x^B(p^A). \end{aligned}$$

Moreover, the choice  $(p^B, q^B)$  of agent  $B$ , for instance, necessarily satisfies the necessary Kuhn-Tucker conditions of his optimization problem, i.e.

$$\begin{aligned} Du^B(e - \tilde{x}^A(p^B, q^B)) &= \lambda_1^B [Du^A(\tilde{x}^A(p^B, q^B)) + D^2u^A(\tilde{x}^A(p^B, q^B))(\tilde{x}^A(p^B, q^B) - e^A)] \\ &\quad + \lambda_2^B Du^A(\tilde{x}^A(p^B, q^B)) \end{aligned}$$

or equivalently, since (2) holds,

$$\begin{aligned} Du^B(e - x^A(p^B)) &= \lambda_1^B D^2u^A(x^A(p^B))(x^A(p^B) - e^A) \\ &\quad + (\lambda_1^B + \lambda_2^B) Du^A(x^A(p^B)) \end{aligned}$$

for some  $\lambda_1^B, \lambda_2^B \geq 0$ , given that at  $x^A(p^B)$  both constraints of  $B$ 's problem equivalently written in terms of allocations

$$\begin{aligned} x^A &\in \max_{\hat{x}^A} u^B(e - \hat{x}^A) \\ Du^A(\hat{x}^A)(\hat{x}^A - e^A) &\geq 0 \\ u^A(\hat{x}^A) &\geq u^A(e - \tilde{x}^B(p^A, q^A)) \end{aligned}$$

are binding. Therefore, since  $Du^A(x^A(p^B))$  is orthogonal to  $(x^A(p^B) - e^A)$  and the Hessian  $D^2u^A(x^A(p^B))$  is negative semidefinite in the space orthogonal to the gradient  $Du^A(x^A(p^B))$  and  $\lambda_1^B \geq 0$ , then

$$\begin{aligned} &Du^B(e - x^A(p^B))(x^A(p^B) - e^A) \\ &= \lambda_1^B(x^A(p^B) - e^A)^t D^2u^A(x^A(p^B))(x^A(p^B) - e^A) \\ &\quad + (\lambda_1^B + \lambda_2^B) Du^A(x^A(p^B))(x^A(p^B) - e^A) \\ &= \lambda_1^B(x^A(p^B) - e^A)^t D^2u^A(x^A(p^B))(x^A(p^B) - e^A) \\ &\leq 0 \end{aligned}$$

that is to say,

$$Du^B(e - x^A(p^B))(x^A(p^B) - e^A) \leq 0.$$

Since also

$$Du^A(x^A(p^B))(x^A(p^B) - e^A) = 0$$

and both  $Du^A(x^A(p^B))$  and  $Du^B(e - x^A(p^B))$  are in  $\mathbb{R}_{++}^2$ , then, without loss of generality, in the case  $A$ 's marginal rate of substitution at  $e^A$  is smaller than  $B$ 's marginal rate of substitution at  $e^B$  (i.e. when  $x_1^A(p^B) - e_1^A < 0$ ),<sup>20</sup>

$$\frac{D_1u^B(e - x^A(p^B))}{D_2u^B(e - x^A(p^B))} \geq \frac{D_1u^A(x^A(p^B))}{D_2u^A(x^A(p^B))}.$$

Similarly, in this same case (where  $x_1^B(p^A) - e^B > 0$ ), from the Kuhn-Tucker conditions of agent  $A$ 's problem follows that

$$\frac{D_1u^A(e - x^B(p^A))}{D_2u^A(e - x^B(p^A))} \leq \frac{D_1u^B(x^B(p^A))}{D_2u^B(x^B(p^A))}$$

But the two previous inequalities can only be satisfied along with the conditions

$$\begin{aligned} u^A(x^A(p^B)) &= u^A(e - x^B(p^A)) \\ u^B(e - x^A(p^B)) &= u^B(x^B(p^A)) \end{aligned}$$

from Lemma 1 only if

$$(x^A(p^B), e - x^A(p^B)) = (x^{A*}, x^{B*}) = (e - x^B(p^A), x^B(p^A))$$

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<sup>20</sup>A similar symmetric argument holds if  $A$ 's marginal rate of substitution at  $e^A$  is bigger than  $B$ 's marginal rate of substitution at  $e^B$  (i.e. when  $x_1^A(p^B) - e_1^A > 0$ ).

and

$$\frac{D_1 u^A(x^{A*})}{D_2 u^A(x^{A*})} = \frac{D_1 u^B(x^{B*})}{D_2 u^B(x^{B*})}.$$

i.e. only if the two allocations coincide and are Walrasian.

In effect, if for instance

$$\frac{D_1 u^B(e - x^A(p^B))}{D_2 u^B(e - x^A(p^B))} > \frac{D_1 u^A(x^A(p^B))}{D_2 u^A(x^A(p^B))},$$

i.e.  $A$ 's marginal rate of substitution at the allocation implicitly proposed by  $B$  is smaller than  $B$ 's marginal rate of substitution, then

$$\frac{D_1 u^A(e - x^B(p^A))}{D_2 u^A(e - x^B(p^A))} > \frac{D_1 u^B(x^B(p^A))}{D_2 u^B(x^B(p^A))}$$

i.e.  $A$ 's marginal rate of substitution at the allocation implicitly proposed by  $A$  is bigger than  $B$ 's marginal rate of substitution because of the convexity of the preferences. This would contradict the inequality above that follows from the Kuhn-tucker conditions of  $A$ 's problem.

A similar argument can be made for any other strict inequality. As a consequence,

$$\begin{aligned} \frac{D_1 u^B(e - x^A(p^B))}{D_2 u^B(e - x^A(p^B))} &= \frac{D_1 u^A(x^A(p^B))}{D_2 u^A(x^A(p^B))} \\ \frac{D_1 u^A(e - x^B(p^A))}{D_2 u^A(e - x^B(p^A))} &= \frac{D_1 u^B(x^B(p^A))}{D_2 u^B(x^B(p^A))}. \end{aligned}$$

i.e. both  $(x^A(p^B), e - x^A(p^B))$  and  $(e - x^B(p^A), x^B(p^A))$  are efficient. This along with the conditions

$$\begin{aligned} u^A(x^A(p^B)) &= u^A(e - x^B(p^A)) \\ u^B(e - x^A(p^B)) &= u^B(x^B(p^A)) \end{aligned}$$

implies the conclusion.

The converse statement holds straightforwardly, since  $\bar{x}^B$  solves the convex program

$$\begin{aligned} \max u^A(e - x^B) \\ u^B(x^B) \geq u^B(e - \bar{x}^A) \end{aligned}$$

and also it satisfies  $Du^B(\bar{x}^B)(\bar{x}^B - e^B) = 0$ . As a consequence,  $\bar{x}^B$  necessarily also solves

$$\begin{aligned} \max u^A(e - x^B) \\ u^B(x^B) \geq u^B(e - \bar{x}^A) \\ Du^B(x^B)(x^B - e^B) \geq 0. \end{aligned}$$

A similar argument holds for  $\bar{x}^A$ . Q.E.D.

**Theorem 2.** *Every allocation of a SSP equilibrium with immediate acceptance converges to a Walrasian allocation as the agents become arbitrarily patient.*

*Proof.* Since every SSP equilibrium with immediate acceptance allocation is Walrasian whenever  $\delta^A, \delta^B = 1$ , the correspondence from discount factors  $(\delta^A, \delta^B)$  to allocations of SSP equilibria with immediate acceptance is compact-valued and upper hemicontinuous, and there are generically finitely many Walrasian allocations, then the conclusion follows.

In effect, consider the correspondence  $\Gamma$  that assigns to each pair of discount factors  $(\delta^A, \delta^B)$  the corresponding allocations of SSP equilibria with immediate acceptance, that is to say such that

$$\Gamma(\delta^A, \delta^B) = \left\{ (x^A, x^B) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 \mid x^A + x^B = e^A + e^B \text{ and} \right. \\ \left. (x^A, x^B) \in \arg \max_{\hat{x}^A} u^B(e - \hat{x}^A) \times \arg \max_{\hat{x}^B} u^A(e - \hat{x}^B) \right\} \\ Du^A(\hat{x}^A)(\hat{x}^A - e^A) \geq 0 \quad Du^B(\hat{x}^B)(\hat{x}^B - e^B) \geq 0 \\ u^A(\hat{x}^A) - \delta^A u^A(e - x^B) \geq 0 \quad u^B(\hat{x}^B) - \delta^B u^B(e - x^A) \geq 0$$

Note that  $\Gamma(\delta^A, \delta^B)$  is the restriction to the feasible allocations of the set of fixed points of the correspondence

$$\Phi(x^A, x^B; \delta^A, \delta^B) = \arg \max_{\hat{x}^A} u^B(e - \hat{x}^A) \times \arg \max_{\hat{x}^B} u^A(e - \hat{x}^B) \\ Du^A(\hat{x}^A)(\hat{x}^A - e^A) \geq 0 \quad Du^B(\hat{x}^B)(\hat{x}^B - e^B) \geq 0 \\ u^A(\hat{x}^A) - \delta^A u^A(e - x^B) \geq 0 \quad u^B(\hat{x}^B) - \delta^B u^B(e - x^A) \geq 0$$

Note also that the Theorem of the Maximum guarantees that

$$\arg \max_{\hat{x}^A} u^B(e - \hat{x}^A) \\ Du^A(\hat{x}^A)(\hat{x}^A - e^A) \geq 0 \\ u^A(\hat{x}^A) - \delta^A u^A(e - x^B) \geq 0$$

is a compact-valued, upper hemicontinuous correspondence that depends on all  $(x^A, x^B, \delta^A, \delta^B)$  – since the function maximizing  $u^B$  depends continuously on  $\hat{x}^A$  and trivially on all  $x^A, x^B, \delta^A, \delta^B$  also and the correspondence defined by the constraints

$$\Omega^A(x^A, x^B, \delta^A, \delta^B) = \{ \hat{x}^A \in \mathbb{R}_+^2 \mid Du^A(\hat{x}^A)(\hat{x}^A - e^A) \geq 0 \text{ and} \\ u^A(\hat{x}^A) - \delta^A u^A(e - x^B) \geq 0 \}$$

is continuous and compact-valued when constrained to the feasible allocations— and similarly for the second arg max.

As a cartesian product of compact-valued, upper hemicontinuous correspondences,  $\Phi$  is compact-valued and upper hemicontinuous itself (see Lemma A1 in Appendix below).

Since

$$\Gamma(\delta^A, \delta^B) = \left\{ (x^A, x^B) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 \mid x^A + x^B = e^A + e^B \text{ and} \right. \\ \left. (x^A, x^B) \in \Phi(x^A, x^B; \delta^A, \delta^B) \right\}$$

and  $\Phi$  is compact-valued and upper hemicontinuous, then  $\Gamma$  is upper hemicontinuous itself (see Lemma A3 in Appendix). Q.E.D.

**Lemma 3.** *Whenever the agents are impatient ( $\delta^A, \delta^B < 1$ ), there does not exist any SSP equilibrium with delay.*

*Proof.* Consider a candidate SSP equilibrium  $(p^A, q^A, p^B, q^B)$  in which, for instance,  $B$  rejects and  $A$  accepts. Let  $x^A = \tilde{x}^A(p^B, q^B)$  and  $x^B = \tilde{x}^B(p^A, q^A)$ . Then it must be the case that

- (1)  $B$ 's offer is rational, that is to say  $B$  is interested in having  $A$  to accept  $x^A$ , i.e.

$$u^B(e - x^A) > \delta^B u^B(x^B)$$

and  $x^A$  is the offer acceptable to  $A$  preferred by  $B$ , i.e.

$$x^A \in \max_{\hat{x}^A} u^B(e - \hat{x}^A) \\ Du^A(\hat{x}^A)(\hat{x}^A - e^A) \geq 0 \\ u^A(\hat{x}^A) \geq \delta^A u^A(e - x^B)$$

(from what follows the rationality of  $A$ 's acceptance) and

- (2)  $A$ 's offer is rational, that is to say  $A$  is interested in having  $B$  to reject  $x^B$ , i.e.

$$\delta^A u^A(x^A) \geq u^A(e - x^B)$$

and accordingly makes an unacceptable offer to  $B$ , i.e.

$$u^B(x^B) < \delta^B u^B(e - x^A)$$

(from what follows the rationality of  $B$ 's rejection).

Therefore, since in such equilibrium  $A$  prefers to make an offer  $x^B$  that will be rejected by  $B$ , i.e.

$$\delta^A u^A(x^A) \geq u^A(e - x^B)$$

and it must be rational for  $A$  to accept  $B$ 's offer  $x^A$ , i.e.

$$u^A(x^A) \geq \delta^A u^A(e - x^B)$$

then, since  $\delta^A < 1$ , necessarily

$$u^A(x^A) > \delta^A u^A(e - x^B).$$

This implies that the second constraint in

$$\begin{aligned} x^A &\in \arg \max_{\hat{x}^A} u^B(e - \hat{x}^A) \\ Du^A(\hat{x}^A)(\hat{x}^A - e^A) &\geq 0 \\ u^A(\hat{x}^A) &\geq u^A(e - x^B) \end{aligned}$$

is not binding and hence  $B$ 's offer  $x^A$  must allow  $B$  to attain the highest possible utility, i.e.  $x^A$  must be  $B$ 's monopoly offer. On the other hand, from the monotonicity of  $u^B$  it follows that necessarily the other constraint must be binding, i.e.  $Du^A(x^A)(x^A - e^A) = 0$ , and hence necessarily  $x^A = x^A(p^A)$ , that is to say  $x^A$  is on  $A$ 's offer curve.

Since we also assume that  $u^A$  is such that at any Walrasian equilibrium allocation the agents' offer curves are transversal to  $A$ 's indifference curve, then  $x^A$  is on  $A$ 's offer curve but not a Walrasian allocation. Therefore,  $(x^A, e - x^A)$  is inefficient. This leaves room for  $A$  deviating profitably offering an  $x^B$  which if accepted by  $B$  (therefore deviating from the candidate equilibrium strategy) makes both  $A$  and  $B$  better-off. Q.E.D.

**Lemma A1.** *If  $X, Y$  are metric spaces and  $\Gamma_1, \Gamma_2 \in \mathcal{P}(Y)^X$  are upper hemicontinuous, then  $\Gamma_1 \times \Gamma_2$  is upper hemicontinuous.*

*Proof.* Assume that  $\Gamma_1 \times \Gamma_2$  is not upper hemicontinuous. Then there exists  $B \subset Y^2$  open such that its upper inverse by  $\Gamma_1 \times \Gamma_2$ , i.e.  $(\Gamma_1 \times \Gamma_2)_+^{-1}(B)$ , is not open. Thus there exists  $x \in (\Gamma_1 \times \Gamma_2)_+^{-1}(B)$ , i.e. satisfying  $\Gamma_1(x) \times \Gamma_2(x) \subset B$ , such that, for all  $\varepsilon > 0$ , there exists  $x' \in B_\varepsilon(x)$  such that  $x' \notin (\Gamma_1 \times \Gamma_2)_+^{-1}(B)$ , i.e.

$$\Gamma_1(x') \times \Gamma_2(x') \not\subset B.$$

Therefore, there exists  $(y_1, y_2) \in \Gamma_1(x') \times \Gamma_2(x')$  such that  $(y_1, y_2) \notin B$ .

On the other hand, since  $\Gamma_1(x) \times \Gamma_2(x) \subset B$  and  $B$  is open, then there exist  $B_1, B_2 \in Y$  open such that (see Lemma A2 below)

$$\Gamma_1(x) \times \Gamma_2(x) \subset B_1 \times B_2 \subset B$$

and hence  $(y_1, y_2) \notin B_1 \times B_2$ , i.e. for some  $i = 1, 2$ ,  $y_i \notin B_i$ . As a consequence, there exists  $B_i \subset Y$  open such that  $\Gamma_i(x) \subset B_i$ , i.e.

$$x \in (\Gamma_i)_+^{-1}(B_i)$$

and, for all  $\varepsilon > 0$ , there exists  $x' \in B_\varepsilon(x)$  such that  $\Gamma_i(x') \not\subset B_i$ , i.e.

$$x' \notin (\Gamma_i)_+^{-1}(B_i).$$

That is to say, for some  $B_i$  open  $(\Gamma_i)_+^{-1}(B_i)$  is not open, which contradicts the assumption that  $\Gamma_i$  is upper hemicontinuous. Q.E.D.

**Lemma A2.** *If  $Y$  is a metric space,  $A_1, A_2 \subset Y$ ,  $B \subset Y^2$ , and*

$$A_1 \times A_2 \subset B,$$

*then there exists  $B_1, B_2 \subset Y$  open such that*

$$A_1 \times A_2 \subset B_1 \times B_2 \subset B.$$

*Proof.* Assume not. Then, for all  $B_1, B_2$  open such that  $A_1 \times A_2 \subset B_1 \times B_2$ ,

$$B_1 \times B_2 \not\subset B.$$

In particular let, for all  $i = 1, 2$  and all  $n \in \mathbb{N}$ ,  $B_i^n = \cup_{\tilde{x}_i \in A_i} B_{\frac{1}{n}}(\tilde{x}_i)$ . Then, for all  $n \in \mathbb{N}$ ,  $B_1^n, B_2^n$  are open and such that

$$A_1 \times A_2 \subset B_1^n \times B_2^n,$$

and hence

$$B_1^n \times B_2^n \not\subset B.$$

Therefore, for each positive integer  $n$  there exists

$$(x_1^n, x_2^n) \in B_1^n \times B_2^n = \cup_{\tilde{x}_1 \in A_1} B_{\frac{1}{n}}(\tilde{x}_1) \times \cup_{\tilde{x}_2 \in A_2} B_{\frac{1}{n}}(\tilde{x}_2)$$

such that  $(x_1^n, x_2^n) \notin B$ . That is to say, for each  $n$  there exist  $(\tilde{x}_1^n, \tilde{x}_2^n) \in A_1 \times A_2$  and  $(x_1^n, x_2^n) \in B_{\frac{1}{n}}(\tilde{x}_1^n) \times B_{\frac{1}{n}}(\tilde{x}_2^n)$  such that  $(x_1^n, x_2^n) \notin B$ . But then, since  $(x_1^n, x_2^n) \in B_{\frac{1}{n}}(\tilde{x}_1^n) \times B_{\frac{1}{n}}(\tilde{x}_2^n)$ , necessarily

$$\lim_{n \rightarrow \infty} d((x_1^n, x_2^n), (\tilde{x}_1^n, \tilde{x}_2^n)) = 0$$

while  $(\tilde{x}_1^n, \tilde{x}_2^n) \in A_1 \times A_2 \subset B$  with  $B$  open, and  $(x_1^n, x_2^n) \notin B$ ! Q.E.D.

**Lemma A3.** *If  $X, Y$  are metric spaces and  $\Phi \in \mathcal{P}(X)^{X \times Y}$  is compact-valued and upper hemicontinuous, then  $\Gamma \in \mathcal{P}(X)^Y$  such that*

$$\Gamma(y) = \{x \in X \mid x \in \Phi(x, y)\}$$

*is upper hemicontinuous.*

*Proof.* Assume that  $\Gamma$  is not upper hemicontinuous at some  $y$ . Then there exist  $\{y_n\} \rightarrow y$ ,  $x$  and  $\{x_n\} \rightarrow x$  such that  $x_n \in \Gamma(y_n)$  for all  $n \in \mathbb{N}$ , while  $x \notin \Gamma(y)$ . That is to say, for all  $n \in \mathbb{N}$ ,  $x_n \in \Phi(x_n, y_n)$  while  $x \notin \Phi(x, y)$ . As a consequence, since  $\Phi$  is compact-valued, then  $\Phi$  is not upper hemicontinuous at  $(x, y)$ ! (in effect, should  $\Phi$  be upper hemicontinuous at  $(x, y)$ , since  $(x_n, y_n) \rightarrow (x, y)$ ,  $x_n \in \Phi(x_n, y_n)$ , and  $\Phi$  is compact-valued, then there would exist a convergent subsequence  $x_{h(n)}$  whose limit  $x'$  would be in  $\Phi(x, y)$ . But since  $\{x_n\}$  itself is convergent to  $x$ , then necessarily  $x' = x$  and hence  $x \in \Phi(x, y)$ !) Q.E.D.

## A Continuum of SSP equilibria exist when $\delta^A, \delta^B = 1$

Consider the extreme case in which both agents are infinitely patient, i.e. when  $\delta^A, \delta^B = 1$ . We show that there is a continuum of SSP equilibria in which, for instance,  $B$  rejects the offer being made while  $A$  accepts.

In effect, let  $A$ 's offer  $x^B$  satisfy the constraint  $Du^B(x^B)(x^B - e^B) \geq 0$  (i.e.  $x^B$  is within  $B$ 's offer curve) and also be such that  $Du^A(e - x^B)(e^A - x^B) > 0$  (in such a way that  $e - x^B$  is strictly within  $A$ 's offer curve) and  $u^A(e - x^B) \geq u^A(\bar{x}^A)$  (i.e.  $A$  is offering an allocation of resources that guarantees him at least his Walrasian equilibrium utility). Let also  $B$ 's offer  $x^A$  maximize  $B$ 's utility subject to the constraint  $Du^A(x^A)(x^A - e^A) \geq 0$  and  $u^A(x^A) \geq u^A(e - x^B)$ , in such a way that  $x^A$  satisfies and  $u^A(x^A) = u^A(e - x^B)$  actually. Note that, on the one hand, since  $x^B$  is within  $B$ 's offer curve and strictly within  $A$ 's offer curve, then necessarily  $u^B(x^B) < u^B(e - x^A)$ , and on the other hand, since  $x^B$  guarantees  $A$  at least  $A$ 's walrasian equilibrium utility,  $x^A$  must lead to an efficient allocation.

Note that for such offers  $x^A$  and  $x^B$ ,

- (1)  $A$ 's offer  $x^B$  is rational since

$$u^A(e - x^B) \leq u^A(x^A)$$

i.e.  $A$  does not prefer  $B$  to accept  $x^B$  given  $B$ 's offer  $x^A$  (in particular this inequality is satisfied with equality), and, by construction,

$$u^B(x^B) < u^B(e - x^A)$$

in such a way that accordingly  $A$  is making an unacceptable offer to  $B$

- (2)  $B$ 's offer  $x^B$  is rational since

$$u^B(e - x^A) > u^B(x^B)$$

and

$$\begin{aligned} x^A &\in \arg \max_{\hat{x}^A} u^B(e - \hat{x}^A) \\ Du^A(\hat{x}^A)(\hat{x}^A - e^A) &\geq 0 \\ u^A(\hat{x}^A) &\geq u^A(e - x^B) \end{aligned}$$

i.e.  $B$  prefers  $A$  to accept  $x^A$  and this is  $B$ 's most preferred offer acceptable to  $A$ , given  $A$ 's offer  $x^B$ , nonetheless,

- (3)  $B$ 's rejection is rational since

$$u^B(x^B) < u^B(e - x^A)$$

and

- (4)  $A$ 's acceptance is rational since

$$u^A(x^A) \geq u^A(e - x^B)$$

Finally, note that, since the allocation eventually agreed upon (i.e. the one resulting from  $A$  accepting  $B$ 's offer  $x^A$ ) is efficient, then there is no room for any profitable deviation by any player.

Note also that, since the choice of  $x^B$  we started with is arbitrary within an uncountable set of  $x^B$ 's satisfying  $Du^B(x^B)(x^B - e^B) \geq 0$ ,  $Du^A(e - x^B)(e^A - x^B) > 0$ , and  $u^A(e - x^B) \geq u^A(\bar{x}^A)$ , then there is uncountably many such equilibria which are moreover efficient.

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