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Sunspots, indeterminacy and Pareto inefficiency in economies with incomplete markets

Tito Pietra[⋆]

Universita' G. D'Annunzio, Chieti-Pescara, Italy

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Abstract We consider two periods economies with both intrinsic and extrinsic uncertainty. Asset markets are incomplete in the certainty economy. If asset are nominal, there are enough commodities and the number of agents is greater than two and smaller than the total number of states of nature "tomorrow" (minus one), then a sunspot-invariant equilibrium is generically Pareto dominated by some sunspot equilibria. When assets are real, and there are enough commodities, if there are sunspot equilibria, there are sunspot equilibria Pareto dominating sunspot-invariant equilibria under the same restriction on the number of agents (and stronger restrictions on the number of commodities).

1 Introduction

Cass (1989) provided the first example of the existence of a continuum of sunspot equilibria in economies with incomplete markets. Since then, existence and structure of sunspot equilibria have been extensively studied in this class of economies with nominal (see, Siconolfi (1989), Cass (1992), Pietra (1992, 2001), Suda, Tallon and Villanacci (1992) and Lisboa (1994)) and with real assets (see, Mas-Colell (1990) and Gottardi and Kajii (1999)). Existence and "number" of sunspot equilibria in financial economies crucially depend upon the nature of the asset payoffs: With nominal assets, they generically exist in a neighborhood of a sunspot-invariant equilibrium

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and their set contains a non-zero dimensional manifold. Its dimension depends upon the way asset prices are treated: Generically, it is $(\Sigma - I)$ if asset prices are fixed,$(\Sigma - 1)$ if they are variable (Σ is the number of states of nature "tomorrow", I the number of assets). When asset payoffs are real (and if there are enough commodities), sunspot equilibria exist for an open set of asset structures and are locally unique.

Our second theme is Pareto dominance. As pointed out in Cass and Shell (1983), in convex economies sunspot equilibria are Pareto inefficient. When asset markets are incomplete at the sunspot-invariant equilibria, they are themselves typically Pareto inefficient and can possibly be Pareto dominated by equilibria of the same economy where extrinsic uncertainty matters. Parametric examples of this phenomenon in incomplete markets economies have been provided in the literature (Pietra (1993) provided an example with $\Sigma = 3$, two agents and one commodity).

Our purpose is to investigate the (comparative) efficiency properties of sunspot and sunspot-invariant equilibria. When asset payoffs are nominal, we show that, generically, sunspot-invariant equilibria are Pareto dominated by some sunspot equilibria, if the number of agents (H) is greater than two and smaller than $(\Sigma - 1)$ and there are enough commodities. We focus the analysis on this comparison. However, an obvious corollary is that, typically, sunspot equilibria are in turn dominated by some other equilibria. A second corollary is that, when $2 < H < (S - 1)$ (S is the number of intrinsic events "tomorrow"), certainty equilibria are typically Pareto dominated by some other certainty equilibria.

The notion of genericity adopted is somewhat weak, because we show that, given a vector of numeraire, there is an open, dense set of economies (parameterized by utility functions and endowments) such that our result holds. A stronger result would be to show that, generically, all the equilibria in a full measure set of sunspot invariant equilibria are Pareto dominated by some sunspot equilibrium. I am not ready to formulate a conjecture on the truthfulness of this result which could possibly be analyzed using the approach exploited in Pietra and Siconolfi (1996) to study real indeterminacy from a global viewpoint in economies with nominal assets and incomplete markets.

Given Magill and Quinzii (1992) (where real indeterminacy of equilibria is re-interpreted in terms of real effects of monetary policies), one possible interpretation of our result is in terms of the existence of Pareto improving randomized monetary policies.

We also study the case of economies with real assets, restricting ourselves to financial structures such that sunspot equilibria exist and with enough commodities. We show that, for a generic set of economies (parameterized by utility functions and endowments), there are asset structures such that sunspot equilibria exist and Pareto dominate at least one of the sunspot-invariant equilibria.

The structure of the paper is the following: Section 2 describes the model. Section 3 establishes that, with nominal assets, sunspot-invariant equilibria are, generically, regular in the sunspot economy and (if they are not Pareto efficient) satisfy some further restrictions on the vector of excess demand and Lagrange multipliers. Then, following the approach outlined by Smale (1974) and developed by Geanakoplos and Polemarchakis (1985), Cass and Citanna (1998) and Citanna, Kajii and Villanacci (1998), we establish that the *extended system of equations* describing the equilibria of the economy and the utility functions of the agents define a system of independent equations (i.e., the derivative of this transformation has maximal rank, equal to the number of equations). This immediately implies that, given a sunspot-invariant equilibrium, there are open sets of Pareto superior sunspot equilibria. While the logic of our argument is the same as in the quoted papers, the technical details are quite different. Those papers establish the full rank property exploiting in a crucial way arbitrary perturbations of the second order derivatives of the utility functions. This approach works in our framework when $(S-1) \geq \mathrm{H}$, it is bound to fail when $\mathrm{H} > (S-1)$, i.e., in the more interesting case when "sunspot matters" in terms of feasibility of Pareto improvements. Therefore, we have to follow a more cumbersome approach.

The last section studies economies with real assets, building on the previous results.

2 The model

We consider a two period model with both intrinsic and extrinsic uncertainty. Spot s = 0 is "today", s = 1, ..., S denote the S intrinsic events "tomorrow". There are also K mutually exclusive extrinsic events, so that, in the sunspot economy, there are KS = Σ states "tomorrow". A generic spot will be denoted either by σ, $\sigma = 0$, (1,1),..., $(1, S), \ldots, (K1), \ldots, (KS)$ or by the more explicit notation ks, when it is important to emphasize the *intrinsic* event characterizing the state of nature.

To simplify notation, and without any loss of generality, we assume that the probability of each extrinsic event is 1/K.

There are H agents, denoted by h = 1,..., H. At each spot there are C commodities, c = 1, ..., C, so that the total number of commodities is $G = ((\Sigma+1)C)$. Agent h' consumption vector at spot σ is $x_h^\sigma = (x_h^{\sigma 1}, \ldots, x_h^{\sigma C})$, while his consumption vector is $x_h = (x_h^0, \ldots, x_h^\Sigma)$. In a similar fashion, excess demand vectors are z_h and z_h^σ, while commodity prices are denoted by $p = (p^0, \ldots, p^\Sigma)$ and $p^\sigma = (p^{\sigma 1}, \ldots, p^{\sigma C})$. Also, let Ψ(p) be the $((\Sigma+1) \times G)$ dimensional matrix

$$\Psi(p) = \begin{bmatrix} p^0 & & \\ & \ddots & \\ & & p^\Sigma \end{bmatrix}.$$

There are I assets. Asset i has payoffs $y^i = [y^{1i}, \ldots, y^{\Sigma i}]$, its price is q^i. Let Y be the $(\Sigma \times I)$ dimensional matrix of asset payoffs and let

$$R(q) = \begin{bmatrix} -q \\ Y \end{bmatrix}$$

be the price-payoffs matrix. For each h, $\mathrm{b}_h \in \Re^I$ is the portfolio vector. The payoffs of asset i can be nominal or can be given, at each state, by the value of a commodity bundle $\rho^{si} = [\rho^{si1}, ..., \rho^{siC}]$.

When assets are nominal, we need three restrictions on asset payoffs:
i. $y^{ks} = y^{k's}$, for each s and k, k';
ii. The matrix $\begin{bmatrix} y^{k1} & \cdots & y^{kS} \end{bmatrix}$ is in general position;
iii. $y^{\sigma 1} = 1$, for each $\sigma > 0$, i.e., asset one is inside money.

Evidently, i guarantees the existence of k-invariant equilibria, iii could be weakened.

For the case of real assets, we will explicitly construct the k-invariant collection ρ^{si}, for each s and i, in Section 4.

Assumptions on consumers are standard: For each h, the endowment vector is $\mathrm{e}_h = (e^0_h, e^{11}_h, ..., e^{SK}_h) \in \Re^G_{++}$, satisfying $e^{sk}_h = e^{sk'}_h$, for each s, k and k'. Given the k-invariance restriction, we will identify the endowment space with $\Re^{(S+1)C}_{++}$. Preferences are described by a Von Neumann-Morgenstern, strictly concave, strictly increasing, C^2, utility function

$$V_h(x_h) = \textstyle\sum_k u_h(x^0_h, x^{k1}_h, ..., x^{kS}_h)/K$$

such that the Bernoulli utility index $\mathrm{u}_h(x^0_h, x^1_h, ..., x^S_h)$ satisfies the usual boundary restriction: the set $\{(x^0_h, x^1_h, ..., x^S_h) \in \Re^{(S+1)C}_{++} | \, u_h(x^0_h, x^1_h, ..., x^S_h) \geq u_h(x^{0*}_h, x^{1*}_h, ..., x^{S*}_h)\}$ is closed in $\Re^{(S+1)C}_{++}$.for each $(x^{0*}_h, x^{1*}_h, ..., x^{S*}_h) \in \Re^{(S+1)C}_{++}$

In the sequel, we will take as given asset payoffs Y (or ρ) and probabilities, π. The space of economies is identified with $\mathcal{E} = \prod_h (\mathcal{U}_h \times \Re^{(S+1)C}_{++})$, where $\mathcal{U}_h$ is the set of Bernoulli utility indexes satisfying the conditions above. As usual, we endow $\mathcal{U}_h$ with the C^2, compact-open topology, $\Re^{(S+1)C}_{++}$ with the Euclidean topology and $\mathcal{E}$ with the product topology. E will denote an element of $\mathcal{E}$

When dealing with nominal assets, we need an appropriate parameterization of the set of equilibria. Here, the most convenient is the following: Fix $p^{\sigma C} = 1$, for each σ, and (locally) parameterize the set of equilibria with a vector $\nu = (1, \nu^{11}, \nu^{12}, ..., 1)$, where we identify ν with its variable coefficients, so that $\nu \in \Re^{\Sigma - 1}_{++}$. Then, the matrix $\Psi(p)$ is replaced by

$$\Psi(p, \nu) = \begin{bmatrix} p^0 & & & \\ & v^{11} p^{11} & & \\ & & \ddots & \\ & & & p^{\Sigma} \end{bmatrix}.$$

In the sequel, $p \in \mathbf{P} = \{p \in \Re^G_{++} | p^{\sigma C} = 1\}$ and $\nu \in \Re^{\Sigma - 1}_{++}$. Also, let $\mathrm{G}^{\setminus} = ((\Sigma + 1)(C - 1))$ and let $p^{\setminus} \in \Re^{G^{\setminus}}_{++}$ be the vector of the first $(C - 1)$ commodity prices at each spot. Finally, let's restrict q to the set of noarbitrage asset prices, $\mathbf{Q} = \left\{ q \in \Re^I | \, \theta R(q) = 0, \text{ for some } \theta \in \Re^{\Sigma + 1}_{++} \right\}$.

Consumer behavior and equilibrium are defined as usual:

Definition 1 *Given* $\nu \in \Re^{\Sigma-1}_{++}$, *an equilibrium is a pair* $(...,(x_h,b_h),....)$ *and* $(p,q) \in \mathbf{P}\times\mathbf{Q}$, *such that:*

i. For each h, $(x_h,b_h) \in \arg\max V_h(x_h)$ *subject to* $\Psi(p,\nu)z_h = R(q)b_h$;

ii. $\sum_h x_h = \sum_h e_h$ *and* $\sum_h b_h = 0$;

iii. An equilibrium is a sunspot equilibrium if, for some h and some s, there are k and k' such that $x^{ks}_h \neq x^{k's}_h$. *Otherwise, it is a k-invariant equilibrium.*

Bear in mind that, to avoid too many repetitions, we use the term sunspot equilibrium to refer only to equilibria where sunspots matter. Equilibria of the sunspot economy where sunspots do not matter are called k-invariant equilibria. Finally, we use the term certainty equilibria to refer to the equilibria of the certainty economy, i.e., (with some abuse of language) of the economy with no *extrinsic* uncertainty.

As well known, k-invariant equilibria are related in an obvious way to certainty equilibria. In particular, $((p^{0*},...,p^{S*}),\ q^*) \in \Re^{(S+1)C+I}$ is a certainty equilibrium associated with $\mu^* \in \Re^{S-1}_{++}$, and with allocation $(...,\ (x^*_h,b^*_h),...)$ and Lagrange multipliers $\lambda^*_h \in \Re^{S+1}_{++}$, for each h, if and only if $((p^{0*},...,(p^{1*},...,p^{S*}),...),q^*) \in \Re^{G+I}$ is a k-invariant equilibrium associated with $\nu = (1,\mu^*,...,\mu^*) \in \Re^{\Sigma-1}_{++}$, allocation $(...,\ ((x^{0*}_h,...,(x^{1*}_h,..., x^{S*}_h),...,b^*_h),...)$ and Lagrange multipliers $(\lambda^{0*}_h,...,(\lambda^{1*}_h/K,...,\lambda^{S*}_h/K),...) \in \Re^{\Sigma+1}_{++}$. We will repeatedly exploit this fact.

In the sequel, we will study equilibria making reference to the entire system of equations implicitly described in Definition 1, after getting rid of the redundant equations, i.e., to the so called *extended system of equations.* Let's define $F(...,(x_h,b_h,\lambda_h),...,(p,q,\nu),...,(u_h,e_h),...)$ as

$$F(\xi,\nu,E) = \begin{bmatrix} ... \\ D_{x_h}V_h(x_h) - \lambda_h\Psi(p,\nu) \\ R(q)^T\lambda_h^T \\ -\Psi(p,\nu)z_h + R(q)b_h \\ .... \\ \sum_h z^{\setminus}_h \\ \sum_h b_h \end{bmatrix},$$

where $z^{\setminus}_h$ denotes agent h's vector of excess demand for all commodities but commodity C at each spot, while $\xi = (...,(x_h,b_h,\lambda_h),...,(p,q))$ and $n = (H(G{+}I{+}\Sigma{+}1){+}G^{\setminus}{+}I)$. $F(.)$ summarizes the first order conditions of each agent and, after applying appropriately Walras' law, the non-redundant market clearing conditions for assets and commodities as a function of the exogenous parameter ν, of the endogenous variables $\xi \in \Re^n$ and of endowments and utility functions. Evidently, F: $\Re^n \times \Re^{\Sigma-1}_{++} \times \mathcal{E} \longrightarrow \Re^n$. Let $F_E(\xi,\nu)$ be the map above for given $E \in \mathcal{E}$: Then, the set of equilibria of E is $F^{-1}_E(0)$.

3 Pareto improving sunspot equilibria with nominal assets

In this section, we consider economies with nominal assets and focus the analysis on the case of variable asset prices.

The basic idea of the proof is standard: The first step is to establish that, for a generic set of economies, k-invariant equilibria are regular and satisfy some additional restrictions (Lemma 1). Then, we show that these equilibria are Pareto dominated by some sunspot equilibria (Theorem 1).

The proof of the theorem requires us to be able to perturb independently the utility functions of two agents at each k-invariant equilibrium. In this class of economies, this is not necessarily possible, due to real indeterminacy. Therefore, we resort to fix $\nu \in \Re^{\Sigma-1}_{++}$: For such a given ν, we will establish the existence of an open and dense set of economies such that their k-invariant equilibria can be Pareto improved by changing appropriately ν.

In the proof of our main theorem, we need to restrict ourselves to regular equilibria where the matrix

$$\left[-Z^{\backslash 1} \cdots - Z^{\backslash K} \ -B \ -W^{\backslash}\right] =$$
$$\begin{bmatrix} \dots & \dots & \dots \\ [\dots, -\lambda_h^\sigma z_h^{\backslash\sigma}, \dots]_{\sigma>0} & [-\lambda_h^0 b_h] & [\dots, -\lambda_h^\sigma p^\sigma z_h^\sigma, \dots]|_{\Sigma>\sigma>0} \\ \dots & \dots & \dots \end{bmatrix},$$

evaluated at the k-invariant equilibrium, has maximal rank H.

Lemma 1 *Let* $H < (S(C-2)+2)$ *and* $I < S$. *Given* $\nu \in \Re^{\Sigma-1}_{++}$, *there is an open and dense subset of* $\mathcal{E}$, $\mathcal{E}^g$, *such that, for each* $E \in \mathcal{E}^g$, *at each k-invariant equilibrium,*

$$rank D_\xi F_E(\xi, \nu)|_{F_E(.)=0} = n,$$
$$rank\left[-Z^{\backslash 1} \cdots - Z^{\backslash K} \ -B \ -W^{\backslash}\right] = H.$$

Proof The first result is basically established in Cass (1992) and Pietra (1992, 2001). The second follows by a routine argument.

As well known, the regularity part of the Lemma holds independently of S (and Σ).

The rank condition obviously implies that the equilibrium is Pareto inefficient. Similar rank conditions are common in the literature on constrained Pareto inefficiency. Our is slightly different (and stronger than the usual one) because we do not consider the $(C-1)$ columns of the excess demands for commodity 0c, c < C. In terms of economic interpretations, bear in mind that the matrix is given by the gradients of the indirect utility functions with respect to $p^{\backslash\sigma}, \sigma > 0$, q and ν.

Given Lemma 1, we can restrict the analysis of the welfare effects of sunspot phenomena to regular sunspot equilibria, satisfying the additional rank condition.

Using the standard approach, consider the system of equations

$$\Phi(\xi, \nu, E) = \begin{bmatrix} F(\xi, \nu, E) \\ V_1(x_1) - V_1^* \\ \dots \\ V_H(x_H) - V_H^* \end{bmatrix}.$$

Suppose that, at the equilibrium ξ (associated with some ν),

$$rank D_{(\xi,\nu)} \Phi_E(\xi, \nu) = (n + H).$$

Then, there exists a vector $(d\xi, d\nu)$ such that

$$D_{(\xi,\nu)} \Phi_E(\xi; \nu) \begin{bmatrix} d\xi \\ d\nu \end{bmatrix} = \begin{bmatrix} 0 \\ \eta \end{bmatrix},$$

for each $\eta >> 0$. Hence, by the implicit function theorem, we can find an appropriate perturbation of ν such that the associated equilibrium allocation Pareto dominates the original one. As mentioned above, our argument is an application of the approach originally introduced by Smale (1974) and developed and applied, in economies with incomplete markets, to issues related to constrained Pareto inefficiency (see, Geanakoplos and Polemarchakis (1986) and Citanna, Kajii and Villanacci (1998)) and to Pareto improving financial innovation (see, Cass and Citanna (1998)).

Theorem 1 *Let* $3 \leq H < (\Sigma - 1)$, $I < S$ *and* $K < (C - 1)$. *Then, for each* $E \in \mathcal{E}^P$, *an open, dense subset of* $\mathcal{E}^g$, *given* $\nu = 1$, *for each k-invariant equilibrium with allocation x' there is a sunspot equilibrium with allocation x" such that, for each h,* $V_h(x_h^{"}) > V_h(x_h^{'})$.

Evidently, $K < (C-1)$ and $H < (\Sigma-1)$ implies that $H < (S(C-2)+2)$, so that Lemma 1 applies. We need at least three agents for technical reasons related to the details of the proof of Lemma 2. It is an open issue if a similar result could be estanblished for H = 2. As an immediate consequence of the theorem, we obtain

Corollary 1 *If* $3 \leq H < (S - 1)$ *and* $I < S$, *then, for each* $E \in \mathcal{E}^P$, *an open, dense subset of* $\mathcal{E}^g$, *given* $\nu = 1$, *for each certainty equilibrium with allocation x' there is a certainty equilibrium with allocation x" such that, for each h,* $V_h(x_h^{"}) > V_h(x_h^{'})$.

Remark 1 Both Theorem 1 and Corollary 1 take as a starting point a regular, k-invariant equilibrium. We can reinterpret the model as follows: Let h denote a type of agents and, therefore, assume that there is a continuum of agents of H different types. Then, our result holds true for economies with a continuum of agents, provided that the characteristics of agents are, a.e., not "too different" from the ones specified in the original H types (see, Cass and Citanna (1998)).

Remark 2 Dealing with certainty equilibria, we do not face additional restrictions (but symmetry, negative semidefiniteness and continuity) on the perturbation of the Hessian matrix of the utility functions. Therefore, the result in Corollary 1 could also be established directly, by a straightforward application of the technique of proof developed starting with Geanakoplos and Polemarchakis (1986) (in fact, with $H \ \geq 2$). However, given the additional restrictions across extrinsic events that we must satisfy in sunspot economies, their technique cannot be applied in the more general framework of Theorem 1.

The proof of Theorem 1 is based on three steps: First, we observe that $rankD_{(\xi,\nu)}\Phi_E(\xi,\nu) = (n+H)$ if and only if

$$rank \begin{bmatrix} D_{(p^{\backslash},q)}\zeta & D_\nu\zeta \\ [-Z^{\backslash} \text{ -B}] & [-W^{\backslash}] \end{bmatrix} = (G^{\backslash} + I + H),$$

where $\zeta(.)$ denotes the aggregate excess demand for all commodities but commodity C at each spot, while, now, $Z^{\backslash}$ is the $(H \times G)$ dimensional matrix with typical coefficient $(-\lambda_h^\sigma z_h^{\sigma c})$, each σ. Next, given $E = (u, e)$ with an equilibrium (p, q), (x, b) and with Lagrange multipliers λ, we pick an economy $\bar{E} = (u, \bar{e})$ such that the initial equilibrium is still an equilibrium of this modified economy and study $rankD_{(\xi,\nu)}\Phi_{\bar{E}}(\xi,\nu)$. We show that, generically in the space of the utility functions, $rankD_{(\xi,\nu)}\Phi_{\bar{E}}(\xi,\nu) = (n+H)$. In fact, to restore its full rank, it suffices to perturb the vector of marginal utilities of two agents. Then, a standard argument shows that (given the new collection of utility functions and modulo a perturbation of the original endowment e in the k-invariant direction $(e-\bar{e})$) $rankD_{(\xi,\nu)}\Phi_E(\xi,\nu) = (n+H)$.

We make precise our argument splitting it into three Lemma.

Let's start displaying $D_{(\xi,\nu)}\Phi_E(\xi,\nu)$,

$$\begin{bmatrix}
\ddots & \cdots & & \cdots \\
& \begin{bmatrix} D^2_{x_h}V_1 & 0 & -\Psi(p,\nu) \\ 0 & 0 & R(q)^T \\ -\Psi(p,\nu) & R(q) & 0 \end{bmatrix} & & \begin{bmatrix} -\Lambda_h^{\backslash}(\nu) & 0 & -\Xi(p,\lambda_h) \\ 0 & -\lambda_h^0 I_I & 0 \\ -\Psi(z_h^{\backslash},\nu) & \begin{bmatrix} -b_h \\ 0 \end{bmatrix} & -W_h^{\backslash} \end{bmatrix} \\
& \vdots & \ddots & \vdots \\
& \begin{bmatrix} I_G^{\backslash} & 0 & 0 \\ 0 & I_I & 0 \end{bmatrix} & & 0 \\
& D_{x_h}V_h & & 0 \\
& \cdots & \ddots & \cdots
\end{bmatrix}$$

where

$$\Psi(z_h^{\backslash},\nu) = \begin{bmatrix} -z_h^{\backslash 0} & & & \\ & \ddots & & \\ & & -\nu^{\Sigma-1} z_h^{\backslash \Sigma-1} & \\ & & & -z_h^{\backslash \Sigma} \end{bmatrix}$$

is a $\left((\varSigma+1)\times G^{\backslash}\right)$ dimensional matrix,

$$W_h^{\backslash} = \begin{bmatrix} 0 & & 0 \\ p^1 z_h^1 & & \\ & \ddots & \\ & & p^{(\varSigma-1)} z_h^{(\varSigma-1)} \\ 0 & & 0 \end{bmatrix}$$

is a $((\varSigma+1)\times(\varSigma-1))$ dimensional matrix, $\varXi(p,\lambda_h)$ is a $((G+I)\times(\varSigma-1))$ dimensional matrix with typical column σ ($\sigma \neq 0, KS$) given by $(0,...,\lambda_h^{\sigma} p^{\sigma},0,...)^T$. Finally, $\varLambda_h^{\backslash}(\nu)$ is the $\left((G+I)\times(G^{\backslash}+I)\right)$ dimensional matrix with all σC rows identically zero. Restricted to the other rows, the matrix is diagonal with coefficients $\lambda_h^{\sigma}\nu^{\sigma}$.

Given that the block diagonal matrices $D_{(x_h,b_h,\lambda_h)}FOC_h$ are readily established to have full rank, $D_{(x,b,\lambda)}FOC^{-1}$ exists. By premultiplying $D_{(\xi,\nu)}\varPhi$ by

$$\begin{bmatrix} -D_{(x,b,\lambda)}FOC^{-1} & 0 \\ 0 & I \end{bmatrix},$$

applying the obvious operations on the rows of the matrix so obtained and, finally, premultiplying by

$$\begin{bmatrix} -D_{(x,b,\lambda)}FOC & 0 \\ 0 & I \end{bmatrix},$$

we conclude that

$$rank D_{(\xi,\nu)}\varPhi = rank \begin{bmatrix} D_{(x,b,\lambda)}FOC & D_{(p^{\backslash},q)}FOC & D_{\nu}FOC \\ 0 & D_{(p^{\backslash},q)}\zeta & D_{\nu}\zeta \\ 0 & \left[-Z^{\backslash}\ -B\right] & -W^{\backslash} \end{bmatrix}.$$

The next step is to compute the rank of the bottom right matrix for an appropriately selected economy $\bar{E}$. Notice that to keep in mind the entire *extended system of equations* and, hence, the matrix $D_{(\xi,\nu)}\varPhi(\xi,\nu,E)$ is essential to keep track of the exact effects that the operation on the columns of $\left[-Z^{\backslash}\ -B\ -W^{\backslash}\right]$ have on the columns of $D_{(p^{\backslash},q,\nu)}\zeta(.)$ via their effects on the columns of $D_{(p^{\backslash},q,\nu)}FOC_h(.)$.

Pick$\nu = 1$ and an associated k-invariant equilibrium (p^*,q^*) with allocation $(...,x_h^*,...)$. We want to construct a new economy $\bar{E}$ with the same equilibrium prices and allocation, but with (k-invariant) individual endowments selected so that they allow to drastically simplify the computations. Bear in mind that prices and allocations are, by assumption, k-invariant, so that we may sometime omit the index k, to emphasize this symmetry.

Let $\hat{s} = min\left\{s|s(C-2)\geq(H-2)\right\}$. The economy $\bar{E}$ is defined as follows: For each h, $\bar{u}_h = u_h$. Endowments are, instead, changed: For agent 1, set $\bar{e}_1^{sc} = x_1^{sc*}$ for all the commodities, but commodity c = 1 at each state s, s > 0, and commodity C at s = 0. Set $\bar{e}_1^{0C} = \left(x_1^{0C*}+q^{1*}\right)$,

$\bar{e}_1^{s1} = \left(x_1^{s1*} - 1/p^{s1*}\right)$. Evidently, given $\bar{e}_1, (x_1^*, (1, 0, ...))$ is agent 1's optimal choice at prices (p*, q*). For agent h, H > h > 1 , define in the obvious way a one to one map

$$f : \{2, ..., H-1\} \rightarrow \{12,\ ...,\ 1\,(C-1)\,,\ 22,\ ...,\ 2\,(C-1)\,, ...,\ \hat{s}(C-1)\}\,,$$

$f(h) = sc$. Then, set $\bar{e}_h^{sc} = x_h^{sc*}$ for each sc but $f(h) = sc$, and commodity C at the same spot. Set $\bar{e}_h^{sc} = (x_h^{sc*} - 1)$, for $sc = f(h)$ and $\bar{e}_h^{sC} = \left(x_h^{sC*} + p^{f(h)*}\right)$. Evidently, given $\bar{e}_h$, $(x_h^*, 0)$ is agent h's optimal choice at prices (p^*, q^*). His excess demand is nil for all commodities, but commodity $sc = f(h)$ and commodity C at the same spot and $z_h^{f(h)*} = 1$. For agent H, set $\bar{e}_H = \left(\sum_h x_h^* - \sum_{h<H} \bar{e}_h^*\right)$.

It is straightforward to check that, given $\nu = 1$, (p^*, q^*) with allocation (x^*) is an equilibrium of the economy $\bar{E}$, supported by the portfolios $b_h = 0$, for $h \neq 1, H$, and $b_1 = -b_H = (1, 0, ...)$.

In words, agent 1 buys one unit of inside money and, at each spot, spends the revenue to buy commodity 1. At each s ≤ ŝ, one unit of each commodity is traded with agent H by agent h such that $sc = f(h)$, who finances the purchase selling commodity C at the same spot. Also, each agent (but 1 and H) trades just once.

Observe that not necessarily $\bar{e} \in \Re_{++}^{(S+1)C}$. However, given that, at $\nu = 1$ and in a neighborhood of (p^*, q^*), for each h and s, $p^{s*}\bar{e}_h^s > 0$, this is irrelevant.

The result driving the entire proof is summarized in the next Lemma. The details are in Appendix. The basic idea is the following: Given $\bar{E}$, the matrix $\left[-Z^{\backslash}\ -B\ -W^{\backslash}\right]$ has a very simple structure. In particular, using columns operations, it can be transformed into a matrix $\left[-Z^{\backslash *}\ 0\ 0\right]$ where $Z^{\backslash 0*} = 0$, while each block $Z^{\backslash k*}$ contains only one nonzero term, in the column corresponding to commodity (ksk). Of course, these columns operations also affect also the matrix $D_{(p^{\backslash}, q, \nu)}\zeta(.)$, transforming it into a matrix $D_{(p^{\backslash}, q, \nu)}\zeta^*(.)$. The key step of the proof is to show that, modulo a perturbation of u_1 and u_2, the $(G^{\backslash} + I)$ columns of $D_{(p^{\backslash}, q, \nu)}\zeta^*(.)$ referred to $(p^{\backslash \circ}, q, \nu)$, where the vector $p^{\backslash \circ}$ does not includes commodity ksk, each k, are linearly independent. When this is true, relabelling columns, we obtain

$$rank \begin{bmatrix} D_{(p^{\backslash}, q)}\zeta & D_{\nu}\zeta \\ \left[-Z^{\backslash}\ -B\right] & -W^{\backslash} \end{bmatrix} = rank \begin{bmatrix} D_{(p^{\backslash \circ}, q)}\zeta^* & D_{\nu}\zeta^* \\ 0 & -Z^{\backslash *} \end{bmatrix} = \left(G^{\backslash} + I + H\right).$$

This sketch of the proof should help to understand why we need a restriction on the number of extrinsic events : We need to replace column $D_{p^{ksk}}\zeta^*$ with column $D_{\nu^{ks}}\zeta^*$, each $ks \neq KS$. This works as long as $K < (C-1)$. The economy $\bar{E}^{\delta}$ constructed in the Lemma depends upon the particular equilibrium we start with. However, for given ν, $F_E^{-1}(0)$ is a finite set (generically). Hence, iterating the procedure, we can show that the same result holds for each equilibrium, associate with the given ν, of an open and dense set of economies.

Lemma 2 *Let* $3 \leq H < (\Sigma - 1)$, $I < \Sigma$ *and* $K < (C-1)$. *Given* ν, $\bar{E}$ *and any open set* $V(\bar{E})$, *there is* $\bar{E}^\delta = (u^\delta, \bar{e}) \in V(\bar{E})$ *with the same equilibrium* (p,q) *and* (x,b) *and such that* $rankD_{(\xi,\nu)}\Phi_{\bar{E}^\delta} = (n+H)$.

Now, let's go back to the actual economy E. Here, we exploit an argument which has been introduced by Balasko (1992) and subsequently exploited, in dealing with sunspot economies, by Cass (1992) and Lisboa (1994).

Lemma 3 *Let* $3 \leq H < (\Sigma - 1), I < \Sigma$ *and* $K < (C-1)$. *Given* ν, E *and any open set* $V(E)$, *there is an open, dense subset* $V\prime(E) \subset V(E)$, *such that, for each* $E^{'} \in V\prime(E)$, $rankD_{(\xi,\nu)}\Phi_{E'}|_{F(.)=0} = (n+H)$.

Proof Start with the economy E and construct $\bar{E}^\delta = (u^\delta, \bar{e})$. By Lemma 2, $rankD_{(\xi,\nu)}\Phi_{\bar{E}^\delta} = (n+H)$. In fact, in Appendix, we establish the result dropping the redundant columns of $D_\nu \Phi_{\bar{E}^\delta}$, so that we obtain a square matrix $D_{(\xi,\nu)}\Phi^{\setminus}_{\bar{E}^\delta}$. Given that $rankD_{(\xi,\nu)}\Phi^{\setminus}_{\bar{E}^\delta} = (n+H)$, $detD_{(\xi,\nu)}\Phi^{\setminus}_{\bar{E}^\delta} \neq 0$. Let $e(t) = (t\bar{e} + (1-t)e)$ and $\bar{E}^\delta(t) = ((u^\delta, e(t))$. Evidently, $detD_{(\xi,\nu)}\Phi^{\setminus}_{\bar{E}^\delta(t)}$ is a polynomial in the variable t. Given that $detD_{(\xi,\nu)}\Phi^{\setminus}_{\bar{E}^\delta(1)} \neq 0$, the polynomial is non trivial and, therefore, $detD_{(\xi,\nu)}\Phi^{\setminus}_{\bar{E}^\delta(t)} = 0$ has a finite number of solutions, $\{t_1, ..., t_T\}$. Hence, if $detD_{(\xi,\nu)}\Phi^{\setminus}_{\bar{E}^\delta(0)} = 0$, it is sufficient to perturb the initial endowment in the (k-invariant) direction $(\bar{e} - e)$ to obtain an (arbitrarily close) economy $E\prime = (u^\delta, e\prime)$ with $detD_{(\xi,\nu)}\Phi^{\setminus}|_{E'} \neq 0$. Hence, $V\prime(E)$ is dense. Given that the initial equilibrium is regular in the economy E, openness of $V\prime(E)$ follows immediately, for $V(E)$ sufficiently small.

We are finally ready to establish Theorem 1.

Proof Density of $\mathcal{E}^P$ follows directly from Lemma 1, 2 and 3. Given the boundary conditions, a standard argument establishes that $\mathcal{E}^P$ is open as well.

4 Pareto improving sunspot equilibria with real assets

As well known, while sunspot equilibria may exist in economies with real assets, they do not exist for arbitrarily given payoffs structures. We will extend the previous result showing that, in economies with incomplete markets (at the k-invariant equilibrium) and such that sunspot equilibria exist, there are, generically, asset payoffs structures such that the original equilibrium is Pareto dominated. To guarantee the existence of sunspot equilibria, we will follow the approach originally proposed by Mas-Colell (1990) and developed in Gottardi and Kajii (1999).

Our construction rests heavily on the previous results. Pick an economy E, nominal asset structure Y and ν' such that the associated equilibrium (more properly, one of the equilibria) (p', q') with allocation (x', b') is Pareto dominated by a sunspot equilibrium associated with ν'', (p'', q''), with allocation (x'', b''). We will show that, typically, there is a k-invariant structure of real assets ρ such that (p', q'), $(x', b')$and (p'', q''), (x'', b'') are both equilibria of the new economy. Notice that, by Corollary 1, if $H < (S - 1)$ and assets are nominal, certainty equilibria are typically Pareto dominated by some other certainty equilibrium, so that the possibility of Pareto improvements within the set of equilibria is not necessarily related to sunspots phenomena. This is in general not true with real assets. It is easy to check that our construction works also for economies with a unique certainty equilibrium.

The proof of the main result exploits an additional property of sunspot equilibria. Define the $((K + 1) \times C)$ dimensional matrix

$$\Xi(s) = \begin{bmatrix} \nu^{s'} p^{s'} \\ \nu^{1s''} p^{1s''} \\ \cdots \\ \nu^{Ks''} p^{Ks''} \end{bmatrix}$$

where the first row is given by state s certainty equilibrium prices at ν', while the last K rows are given by equilibrium prices at the states (ks) at ν''.

Lemma 4 *Let* $3 \leq H < (\Sigma - 1)$, $I < S$ *and* $K < (C - 2)$. *Given the nominal asset structure Y, let* (p', q') *be the k-invariant equilibrium associated with* ν' *and* (p'', q'') *be a sunspot equilibrium associated with* ν'', *satisfying* $v^{k_a s''} \neq v^{k_b s''}$ *for each s and each pair* k_a, k_b. *Then, for an open and dense set* $\mathcal{E}^* \subset \mathcal{E}$, $\Xi(s)$ *has maximal rank* $(K + 1)$, *for each s.*

Proof By Lemma 1, there is no loss of generality in assuming that both k-invariant and sunspot equilibria are regular and sufficiently close (in the economy with nominal assets). Hence, it suffices to show the density part of the thesis, i.e., that for each open ball $V(E)$, there is $E^* \in V(E)$ such that $rank \Xi(s) = (K + 1)$, for each s, at the equilibrium (p^*, q^*). By a standard argument, modulo a perturbation of the utility function in the certainty economy, there is an h, say h = 1, such that, at each σ, $p^{\sigma'} z_1^{\sigma'} \neq 0$. By regularity of the sunspot equilibrium and continuity, we can assume that $\mathrm{p}^{\sigma''} z_1^{\sigma''} \neq 0$, at each σ. Given that $v^{k_a s''} \neq v^{k_b s''}$, each s and each pair k_a, k_b, this implies $x_1^{k_a s''} \neq x_1^{k_b s''}$, each s and each pair k_a, k_b, and $\mathrm{x}_1^{s'} \neq x_1^{ks''}$, each k. Hence, there exists a collection of K open balls $B_1(x_1^{k''})$, each one centered on $x_1^{k''} = (x_1^{0''}, x_1^{k1''}, ..., x_1^{kS''})$, k = 1, ..., K, and a collection of closed balls $B_1^{\varepsilon}(x_1^{k''}) \subset \Re_{++}^{(S+1)C}$ such that $B_1(x_1^{k''}) \subset B_1^{\varepsilon}(x_1^{k''}) \subset \Re_{++}^{(S+1)C}$, each k, and $B_1^{\varepsilon}(x_1^{k_a''}) \cap B_1^{\varepsilon}(x_1^{k_b''}) = \emptyset$, each k_a, k_b and $x_1' \notin B_1^{\varepsilon}(x_1^{k''})$, each k. Also, let $\Theta^k(x_1, B_1^{\varepsilon}(x_1^{k''}))$, $k = 1, ..., K$, be a collection of bump functions taking the value 1 at $\mathrm{x}_1 \in B_1(x_1^{k''})$, the value 0 at $\mathrm{x}_1 \notin B_1^{\varepsilon}(x_1^{k''})$. Assume that $rank \Xi(s) < (K + 1)$, for some

s. Consider an arbitrarily small perturbation of its last K rows (keeping $p^{ks1} = 1$, each k), so that the new matrix $\Xi'(s)$ has full rank. Repeat the operation for each s > 0. Let p* be the price vector so obtained and let $\Delta^*(p^*, q^*, \nu") = \sum_{h>1} z_h^{\sigma}(p^*, q^*, \nu")$ and $z_1^* = z_1(p^*, q^*, \nu")$. Given that $\sum_h z_h^{\sigma}(p", q", \nu") = 0$, by continuity, $|| - \Delta(p^*, q^*, \nu") - z_1(p^*, q^*, \nu")||$ can be made arbitrarily close to 0 by choosing a price vector p^* sufficiently close to $p"$ and still preserving the full rank of $\Xi'(s)$, each s. Consider now the economy E^* with $e_h^* =$ e$_h$, each h, $u_h^* = u_h$, h > 1, and $u_1^*(x_1) = u_1(x_1) + \sum_k \Theta^k(x_1, B_1^{\varepsilon}(x_1^{k"}))[D_{x_1} u_1(x_1)|_{z_1^*} - D_{x_1} u_1(x_1)|_{-\Delta^*}]x_1$. I claim that, $(-\Delta^*, b_1^* = -\sum_{h>1} b_h(p^*, q^*, \nu"))$ is agent 1's optimal choice given (p^*, q^*), $\nu"$ and utility function u_1^*. Evidently, $(-\Delta^*, b_1^*)$ satisfies agent 1's sequence of budget constraints. Moreover, given that, with utility u$_1$, at z_1^*, $\sum_k \pi(k) D_{x_1} u_1(x_1)|_{z_1^*} = \lambda_1^{*T} \Psi(p^*, \nu")$, for some vector λ^* such that $\lambda_1^{*T} R(q^*) = 0$, $\sum_k \pi(k) D_{x_1} u_1^*(x_1^k)|_{-\Delta^*} = \sum_k \pi(k) D_{x_1} u_1(x_1^k)|_{-\Delta^*} + \sum_k \pi(k)[D_{x_1} u_1(x_1)|_{z_1^*} - D_{x_1} u_1(x_1)|_{-\Delta^*}] = \sum_k \pi(k)[D_{x_1} u_1(x_1)|_{z_1^*} = \lambda_1^{*T} \Psi(p^*, \nu")$. Hence, $(-\Delta^*, b_1^*)$ solves the FOC of agent 1's optimization problem at (p^*, q^*) and $\nu"$. By construction, the allocation $(z_1 = -\Delta^*, z_2(p^*, q^*, \nu"),$ ..., $z_H(p^*, q^*, \nu"))$ satisfies the market clearing conditions. Given that Y is a full rank matrix, this means that the market clearing conditions for assets are satisfied as well. Hence, (p^*, q^*) is a sunspot equilibrium of the economy E^*, given $\nu"$. Moreover, given that agent 1's utility function is not changed at x$_1^{'}$, the k-invariant equilibrium is not affected. As noticed above, for a sufficiently small perturbation of the equilibrium prices $p"$, E^* can be made arbitrarily close to E, hence $E^* \in V(E)$.

Finally,

Theorem 2 *Let $3 \leq H < (\Sigma - 1)$, $I < S$ and $K < (C - 2)$. Then, there exists an open, dense set of economies $\mathcal{E}^p \subset \mathcal{E}$ such that, for each $E \in \mathcal{E}^P$, there is an asset structure ρ such that there are sunspot equilibria Pareto superior to a k-invariant equilibrium.*

Proof Given Lemma 4, it suffices to establish the density part. Given E, any open set $V(E)$ and any asset structure Y, with I < S, pick E^* such that the regular k-invariant equilibrium associated with ν is Pareto dominated by a sunspot equilibrium associated with $\nu"$. Given the previous Lemma, we may assume that, at the sunspot equilibrium $p"$, $rank \Xi(s) = (K + 1)$, each s. This implies that, for each s and each i, there is a solution $\rho^{i*}(s)$ to the system of equations $\Xi(s)\rho^i(s) = [y^{is}, y^{is}, ..., y^{is}]$. Then, let ρ^* define the structure of real asset of the economy. Evidently, (p', q') is a k-invariant equilibrium, while $(p", q")$ is a sunspot equilibrium of the new economy.

Remark 3 In the economy with real assets, neither k-invariant nor sunspot equilibria are necessarily regular (because $D_{(p,q)}\zeta(.)$ in the economy with real assets is obviously different from the same derivative in the economy with nominal assets). Given Gottardi and Kajii (1999), our result can be strenghtened by considering equilibria which are regular in the economy with real assets, too. This would allow to strenghten Theorem 2, establishing openness of the set of asset structures allowing for Pareto improvements. The argument is straightforward and, therefore, omitted.

5 Appendix: Proof of Lemma 2

Throughout the proof we keep fixed the equilibrium (p, q) and $\nu = 1$ and perturb the economy $\bar{E}$ so that $(p, q) \in F_{\bar{E}'}^{-1}(0)$. The proof requires us to be able to perturb (in k-invariant directions) the gradients of three agents. The argument is standard and reported here for completeness.

Claim Let $I < S$. Fix ν^*, $\bar{E}$, any open set $V(\bar{E})$ and a k-invariant equilibrium (p^*, q^*), (x^*, b^*) with associate Lagrange multipliers λ^*. Then, for each $\delta \in \Re^{\Sigma+1}$, δ small enough and such that $R(q)\delta^T = 0$, there is $\bar{E}^\delta \in V(\bar{E})$ such that (p^*, q^*), (x^*, b^*) is an equilibrium of $\bar{E}^\delta$ with associated Lagrange multipliers $\lambda_h^\delta = (\lambda_h^* + \delta)$.

Proof Consider the certainty equilibrium (p^*, q^*), associated with the k-invariant equilibriumm and the associated vector $(x_h^*, b_h^*, \lambda_h^*)$. Consider agent h's utility function $u_h(x_h)$ and any open ball $V(u_h)$. Pick an open ball $B_h(x_h^*) \in \Re_{++}^{(S+1)C}$ and $\varepsilon > 0$, ε small enough, so that the closed set $B_h^\varepsilon(x_h^*)$ satisfies: $B_h(x_h^*) \subset B_h^\varepsilon(x_h^*) \subset \Re_{++}^{(S+1)C}$. Let $\Theta_h(x_h, B_h^\varepsilon)$ be a smooth function taking the value 1 on $B_h(x_h^*)$, the value 0 for $x_h \notin B_h^\varepsilon(x_h^*)$ (see, Mas-Colell (1986)). Set $u_h^\delta(x_h) = u_h(x_h) + \Theta_h(x_h, B_h^\varepsilon)[\sum_s \delta_h^s \nu^{s*} p^{s*} x_h^s)]$, for δ_h small enough and such that $R(q)\delta_h^T = 0$. Evidently, $D_{x_h} u_h^\delta(x_h^*) - (\lambda_h^* + \delta_h)\Psi(p^*, \nu^*) = 0$, $R(q)(\lambda_h^* + \delta_h)^T = 0$ and $\Psi(p^*, \nu^*)z_h^* - R(q) = 0$. Hence, (p^*, q^*), with associated (x_h^*, b_h^*) is an equilibrium of the economy $\bar{E}^\delta = \prod_h (u_h^\delta, \bar{e}_h)$. By construction, h's vector of Lagrange multipliers is $(\lambda_h^* + \delta_h)$.

The argument is based on two steps:

1. We exploit the k-invariance of the allocation to get rid of all but H linearly independent columns of the submatrix [Z$^\backslash$ B W$^\backslash$]. This may require a perturbation of agent H's utility function.

2. We show that, modulo an arbitrarily small perturbation of u_2, the $(G^\backslash + I)$ dimensional, square matrix $D_{(p^{\backslash\circ}, q, \nu)}\zeta$, defined above, has full rank.

The Lemma follows immediately.

Step 1

Bear in mind that $K < (C - 1)$ and $H < (\Sigma - 1)$.

Without loss of generality, but for notational simplicity, fix $\nu^{ks} = 1$ for each s $> \hat{s}$ and drop the corresponding columns of W$^\backslash$ and $D_\nu \zeta$ (remember that $\hat{s} = min\{s | s(C - 2) \geq (H - 2)\}$).

Also for notational convenience, let $\ell_h = K(\lambda_h^{11}, ..., \lambda_h^{KS})/\lambda_h^0$. Hence, if $\ell_h^c = (\ell_h^1, ..., \ell_h^S)$ is the vector of period 1, normalized (by λ_h^0) Lagrange multipliers at the certainty equilibrium, $\ell_h = (..., \ell_h^c, ...)$ is the normalized (by λ_h^0/K) period 1 vector of Lagrange multipliers at the associate k-invariant equilibrium.

In the economy $\bar{E}$ (and after normalizing row h by λ_h^0/K, each h), Z$^\setminus$ is given by $\left[\, KZ^{\setminus 0}\ Z^{\setminus 1}\ ...\ Z^{\setminus K} \,\right]$, with Z$^{\setminus 0}$ = [0], while $-Z^{\setminus k} =$

$$\begin{bmatrix}
-\ell_1^1/p^{11} & & & -\ell_1^2/p^{21} & & & & & \cdots & -\ell_1^S/p^{S1} \\
 & \ddots & & & & & & & & \\
 & & -\ell_{C-1}^1 & & & & & & & \\
 & & & & -\ell_C^2 & & & & & \\
 & & & & & -\ell_{C+1}^2 & & & & \\
 & & & & & & \ddots & & & \\
 & & & & & & & -\ell_{H-1}^{\hat{s}} & \cdots & \\
\ell_H^1/p^{11} & & \ell_H^1 & \ell_H^2/p^{21} & \ell_H^2/p^{22} & \ell_H^2 & & \ell_H^{\hat{s}} & & \ell_H^S/p^{S1}
\end{bmatrix},$$

and

$$\left[\, -(KB)\ -W \,\right] = \left[\begin{bmatrix} -K & 0 \\ \cdots & \cdots \\ K & 0 \end{bmatrix} \begin{bmatrix} -\ell_1^1 & & -\ell_1^{\hat{s}} \\ & \ddots & \\ \ell_H^1 & & \ell_H^{\hat{s}} \end{bmatrix} \right].$$

Evidently, there is no loss of generality in assuming that Z$^\setminus$ has maximal rank H: Relabelling spots and agents, it suffices to pick agent 1 and H so that $\ell_1^s/\ell_1^{s'} \neq \ell_H^s/\ell_H^{s'}$, for some s and s'. With reference to the previous claim, set $\delta_H = \left[\delta_H^0, \delta_H^1, 0, \delta_H^3, ..., \delta_H^S\right]$, $\delta_H^1 \neq 0$, and δ_H^s, $s \neq 1, 2$, chosen so that $R(q)\delta_H = 0$, which can always be done because $\left[\, y^{k1} \cdots y^{kS} \,\right]$ is in general position, for each k.

Consider the following sequence of column operations (we identify columns in the obvious way), focusing, for the moment, on their effects on the columns of $\left[\, Z^\setminus\ (KB)\ W^\setminus \,\right]$:

a. For each s, ŝ ≥ s > 0, subtract from column (ν^{ks}) column (p^{ks1}) multiplied by p^{s1} (i.e., get rid of the nonzero coefficients of the matrix $W^\setminus$).

b. Subtract column (p^{ksc}), $c =$ k, from column $(p^{k'sk})$ to eliminate all the (collinear) columns $(p^{k'sk})$, k' $\neq k$.

c. Use the linearly independent columns (p^{111}) and (p^{121}) to eliminate columns (p^{1s1}), $s > 2$, and the nonzero columns of the matrix B.

Step 2:

Rearrange the columns of the (transformed) matrix to obtain

$$\begin{bmatrix} D_{(p^{\setminus\circ}, q, \nu)}\zeta^* & D_{p^\circ}\zeta^* \\ 0 & Z^* \end{bmatrix},$$

where Z^* has full rank H. The square matrix $D_{(p^{\setminus\circ}, q, \nu)}\zeta^*$ is given by the columns (modified by the operations under a, b and c above) relative to (p^{0c}), $c < C$, (p^{ksc}), each ksc $\neq ksk$ and $\mathrm{c} < \mathrm{C}$, and (q^{i}). The other H columns relative to (p^{ksk}), $\mathrm{s} \leq \hat{\mathrm{s}}$, have been replaced by the columns (ν^{ks}).

We need to show that, generically, rank $D_{(p^{\setminus\circ},q,\nu)}\zeta^* = \left(G^{\setminus} + I\right)$.

Let $[D_{(x_h,b_h,\lambda_h)}FOC_h]^{\setminus -1}$ be the matrix obtained from $[D_{(x_h,b_h,\lambda_h)}FOC_h]^{-1}$ deleting the rows associated with commodity σC, each σ, and the last $(\Sigma{+}1)$ rows. By the implicit function theorem,

$$D_{(p^{\setminus},q,\nu)}\ z_h^{\setminus} = -[D_{(x_h,b_h,\lambda_h)}FOC_h]^{\setminus -1}D_{(p^{\setminus},q,\nu)}FOC_h.$$

Let $D_{(p^{\setminus\circ},q,\nu)}FOC_h^*$ be the matrix obtained from$D_{(p^{\setminus},q,\nu)}FOC_h$ with the operations performed in step 1 and the substitution of columns just described. Then, $D_{(p^{\setminus\circ},q,\nu)}\zeta^* = -\sum_h[D_{(x_h,b_h,\lambda_h)}FOC_h]^{\setminus -1}D_{(p^{\setminus\circ},q,\nu)}FOC_h^*$.

Claim Assume that $[D_{(x_h,b_h,\lambda_h)}FOC_h]^{\setminus -1}D_{(p^{\setminus\circ},q,\nu)}FOC_h^* = \lambda_h^0 M_h$, where M_h is a square $(G^{\setminus} + I)$ dimensional matrix of full rank, which does not depend upon λ_h^0. Then, given any open set $V(\bar{E})$, there is $\bar{E}^1 \in V(\bar{E})$ such that $(p,q) \in F^{-1}(0)$ and $rankD_{(p^{\setminus\circ},q,\nu)}\zeta^*|_{\bar{E}^1} = (G^{\setminus} + I)$.

Proof As established above, by choosing an appropriate perturbation of u_h we can arbitrarily perturb λ_h^0 , without affecting the equilibrium. With reference to the previous claim, simply set $\delta_2 = \lambda_2\eta$. It is easy to see that agent 2's vector of Lagrange multipliers becomes $\lambda_2(1-\eta)$: Hence, the normalized vector does not change, while $\lambda_2^{0\delta} = \lambda_2^0(1-\eta)$.

Then, $D_{(p^{\setminus\circ},q,\nu)}\zeta^*|_{\bar{E}^\delta} = D_{(p^{\setminus\circ},q,\nu)}\zeta^*|_{\bar{E}} - (\eta\lambda_2^0)M_2$ and, therefore, $(\eta\lambda_2^0)$ is an eigenvalue of the matrix $M_2^{-1}D_{(p^{\setminus\circ},q,\nu)}\zeta^*|_{\bar{E}}$. The set of eigenvalues of any matrix is finite, hence, for almost all η, $rankD_{(p^{\setminus\circ},q,\nu)}\zeta^*|_{\bar{E}^\delta} = (G^{\setminus} + I)$.

Therefore, to conclude, we just need to show $rankM_h = \left(G^{\setminus} + I\right)$, for some h. Consider h = 2 and let

$$[D_{(x_2,b_2,\lambda_2)}FOC_2]^{-1} = \begin{bmatrix} \Delta_2^{1*} & \Delta_2^{2T} \\ \Delta_2^2 & \Delta_2^3 \end{bmatrix},$$

where Δ_2^{1*} is well known to be a $(G + I)$ dimensional square matrix of rank $(G^{\setminus}{+}I)$. Also, let Δ_2^1 be the $\left((G^{\setminus} + I) \times (G + I)\right)$ dimensional matrix obtained deleting rows σC, each σ. It is well-known that $rank\Delta_2^1 = (G^{\setminus}{+}I)$ (see, for instance, Balasko and Cass (1991)). Moreover, $\Delta_2^1\left[-\Psi(p,\nu)\ R(q)\right]^T = 0$, so that $\left[-\Psi(p,\nu)\ R(q)\right]^T$ spans the null space of Δ_2^1. To conclude the proof, first we observe that the last $(\Sigma + 1)$ rows of $D_{(p^{\setminus\circ},q,\nu)}FOC_2^*$ are identically zero and that $D_{(p^{\setminus\circ},q,\nu)}FOC_2^*$ is homogeneous of degree 1 in λ_2^0, so that

$$D_{(p^{\setminus\circ},q,\nu)}FOC_2^* = \begin{bmatrix} \lambda_2^0 D_2 \\ 0 \end{bmatrix}$$

and, therefore, $[D_{(x_1,b_1,\lambda_1)}FOC_2]^{\setminus -1}D_{(p^{\setminus\circ},q,\nu)}FOC_2^* = \lambda_2^0\Delta_2^1 D_2$. Finally, we show that:

a. D_2 has full column rank;

b. $spanD_2 \cap span\left[-\Psi(p,\nu)\ R(q)\right]^T = \{0\}$.

It follows that $B_2^1 \Delta_2 = M_2$ has full rank $\left(G^\setminus + I\right)$, as required.

Bear in mind that, at a k-invariant equilibrium, p^{ks} and ℓ_2^{ks} are k-invariant, so that we can omit the index k, to stress the symmetry. Consider the changes induced in the matrix $D_{(p^\setminus,q,\nu)}FOC_2$ by the sequence of operations on the columns and by their final rearranging. The first $(G^\setminus + I)$ columns of the modified matrix (using the normalization by K/λ_2^0) are given by the matrix $(\lambda_2^0/K)D_2^*$, whose last $(\Sigma + 1)$ rows are identically zero (in step 1, we got rid of all the non-zero terms of these rows but $-z_2^{11}$ and $-z_2^{21}$ and the two corresponding columns have been replaced by the ones associated with ν^{11} and ν^{12}). Indeed,

$$D_2^* = \left[\begin{array}{cccccccccc}
-KI^\setminus & & & & & & & & & \\
 & -A^{11} & & & C_S^{11} & & T^{11} & & & C_I^{11} \\
 & & -A^{12} & & C_S^{12} & & & & & C_I^{12} \\
 & & & \ddots & & & & & & \\
 & & & & -A^{1S} & & & & T^{1S} & \\
 & & & & & \ddots & & & & \\
 & T^{K1} & & & & & -A^{K1} & & & \\
 & & & & & & & \ddots & & \\
 & & & & & & & & -A^{KS} & \\
 & & & & & & & & & -KI_I \\
0 & \cdots & & & & & & & \cdots & 0
\end{array}\right].$$

Where:

$$A^{ks} = \left[\begin{array}{ccccc}
\ell_2^s & & 0 & & \\
 & \ddots & & & \\
 & & \ell_2^s p^{ksk} & & \\
 & & & \ddots & \\
 & & & & \ell_2^s \\
 & & \ell_2^s & & 0
\end{array}\right],$$

i.e., the vector $\left[0\ \ell_2^s p^{s2} \cdots \ \ell_2^s\right]^T$ replaces the column referred to commodity c = k (remember that we subtracted from column ν^{ks} column (p^{1s1}) multiplied by p^{s1} in step 1a and, then, replaced column (p^{ksk}) with the (modified) column ν^{ks}).

The matrices T^{ks}, $s \leq \hat{s}$, are given by all zero coefficients, but the coefficient corresponding to commodity c = k, which is equal to ℓ_1^s (remember step 1b above).

The matrices C_s^{1t} and C_I^{1t}, t = 1, 2, s > 2, reflect the operations of step 1c. Then, for s > 2, t = 1, 2, C$_s^{1t}$ has all the coefficients equal to zero, but the ones on the first row and column, given by $\ell_2^t c_s^{1t}$, where

$$\begin{bmatrix} c_s^{11} \\ c_s^{12} \end{bmatrix} = \begin{bmatrix} -\ell_1^1/p^{11} & -\ell_1^2/p^{21} \\ \ell_H^1/p^{11} & \ell_H^2/p^{21} \end{bmatrix}^{-1} \begin{bmatrix} -\ell_1^s/p^{s1} \\ \ell_H^s/p^{s1} \end{bmatrix}.$$

Similarly, C_I^{1t}, $t = 1, 2$, has all the coefficients equal to zero, but the ones on the first row and column, given by $\ell_2^t c_I^{1t}$,where

$$\begin{bmatrix} c_I^{11} \\ c_I^{12} \end{bmatrix} = \begin{bmatrix} -\ell_1^1/p^{11} & -\ell_1^2/p^{21} \\ \ell_H^1/p^{11} & \ell_H^2/p^{21} \end{bmatrix}^{-1} \begin{bmatrix} -K \\ K \end{bmatrix}.$$

Bear in mind that these matrices just appear on the blocks of rows of states 11 and 12, given the asymmetric role played by the value of the excess demand in these two states.

Let D_2 be the $\left((G+I) \times (G^\backslash + I)\right)$ dimensional matrix given by the first $(G+I)$ rows of D_2^*. Given that $\nu = 1$ and is fixed, for the purpose of this computation, we can drop it from the notation.

Claim D_2 has full column rank.

Proof Suppose that $D_2 a^T = 0$, and let $a = (a^0, a^1, ..., a^K, a^{\Im})$, with $a^{\Im} = (a^{\Im 1}, ..., a^{\Im I})$. Evidently, $a^0 = 0$, $a^{\Im} = 0$ and $a^{ks} = 0$, for each $\mathrm{s} > \hat{\mathrm{s}}$. For a^{ks}, $\mathrm{k} > 1$, given the structure of A^{ks}, $a^{ksk} = 0$, because for each ksk there is a row (the one corresponding to commodity ksC) with one and only one non zero coefficient. This immediately implies that $a^{ksc} = 0$, each c. For $\mathrm{k} = 1$, the same argument applies to blocks ks, $\mathrm{s} > 2$. Taking into account these facts and given that A^{11} has full rank, $a^{11} = a^{12} = 0$.

Claim Let $I \ < S$, and $3 \leq H$. Then, for each open set $V(\bar{E})$, there is an open and dense subset $V\prime(\bar{E}) \subset V(\bar{E})$ such that, for each $\bar{E}' \in V'(\bar{E})$, $span D_2 \cap span \left[-\Psi(p)^T \ R(q)^T \right] = \{0\}$.

Proof Openness follows by regularity of the equilibrium (if $V(\bar{E})$ is sufficiently small). Hence we just show the density part.

If $b \in span \left[-\Psi(p)^T \ R(q)^T \right]^T$, then
$b = \left[-b^0 p^0, (-b^{11} p^{11}, ..., -b^{1S} p^{1S}), ..., (-b^{K1} p^{11}, ..., -b^{KS} p^{1S}), b^{\Im} \right]$,
with $b^{\Im} = R(q)^T \left[b^0, ..., b^K \right]^T$. The structure of b imposes several restrictions on the coefficients of the vector a. We are going to show that, generically in u_2, they are satisfied if and only if $a = 0$, establishing the claim.

Let $a = \left[a^0, a^1, a^K, a^{\Im} \right]$, as above, and assume that $b = D_1 a$, for some a and $b \in span \left[-\Psi(p)^T \ R(q)^T \right]^T$.

Given that $-b^0 p^0 = -\left[a^0, 0 \right]$, it must be $a^0 = b^0 = 0$.

Consider states 11 and 12. Given that, for $t = 1, 2$, $-b^{1t} p^{1t} =$
$\left[\ell_2^t \left(a^{\Im 1} c_1^{1t} + \sum_{s>2} a^{1s1} c_1^{1t} + \sum_{k>1} a^{kt1} \right), ..., -(\ell_2^t a^{1t1} p^{tc} + \ell_2^{\mathrm{t}} a^{1tc}), ..., -\ell_2^{\mathrm{t}} a^{1t1} \right]$,
it must be $a^{1tc} = 0, C > c > 1$, and $\left(a^{\Im 1} c_1^{1t} \ + \sum_{s>2} a^{1s1} c_1^{1t} + \sum_{k>1} a^{kt1} \right) = -a^{1t1} p^{t1}$, so that $b^{1t} = \ell_2^t a^{1t1}$.

For b^{1s}, $\hat{\mathrm{s}} \geq \mathrm{s} > 2$, given that
$-b^{1s} p^{1s} = \left[\ell_2^s \sum_{k>1} a^{ks1}, -(\ell_2^{\mathrm{s}} a^{1s1} p^{s2} + \ell_2^{\mathrm{s}} a^{1s2}), ..., -\ell_2^{\mathrm{s}} a^{1s1} \right]$,

$a^{1sc} = 0$, $c > 1$, and $\sum_{k>1} a^{ks1} =$ -$a^{1s1}p^{s1}$, so that $b^{1s} = \ell_2^s a^{1s1}$.

For k > 1, and s $\leq \hat{s}$, given that $-b^{ks}p^{ks} = -\left[\ell_2^{\mathrm{s}} a^{ks1}, .., \ell_2^s a^{ksk}p^{sc}, ..., \ell_2^{\mathrm{s}} a^{ksk}\right] + \left[0, -\ell_2^s a^{ks2}, ..., \ell_2^s \left(\sum_{k' \neq k} a^{k'sk}\right), ..., -\ell_2^s a^{ksC-1}, 0\right]$, it must be $a^{ksc} = 0$, for $c \neq 1$, k, $a^{ks1} = a^{ksk}p^{s1}$, $\sum_{k' \neq k} a^{k's} = 0$ and $b^{ks} = \ell_2^s a^{ksk}$.

For s $\geq \hat{s}$, given that $-b^{ks}p^{ks} =$ -$\ell_1^s(a^{ks}, 0)$, it must be $a^{ks} = b^{ks} = 0$, each k.

Given that $a^{ksc} = 0$, for all c, but c $= 1$, k (each s and k), and that $a^{ks1} = a^{ksk}p^{s1}$, we can denote a^{ksk} as a^{ks}.

Finally, given that $-K\left[a^{\Im}\right]^T = b^{\Im T} = R(q)^T\left[b^0, ..., b^K\right]^T$ and taking into consideration what we have just established, it must be $-K\left[a^{\Im}\right]^T = R(q)^T\left[0 \; \ell_2^1 a^{11} \cdots \; \ell_2^{\mathrm{S}} a^{KS}\right]^T$. To summarize, we must have:

a. $-K\left[a^{\Im}\right]^T = R(q)^T\left[0 \; \ell_2^1 a^{11} \cdots \; \ell_2^S a^{KS}\right]^T$;

b. $a^0 = a^{ks} = 0$, for s $>$ ŝ and each k;

c. $\sum_{k' \neq k} a^{k's} = 0$, k > 1 and each s;

d. $\sum_{k>1} a^{ks} = -a^{1s}$, s > 2;

e. $\left(a^{\Im 1}c_1^{1t}/p^{t1} + \sum_{s>2} a^{1s}c_1^{1t}/p^{t1} + \sum_{k>1} a^{kt}\right) = -a^{1t}$, t $= 1$, 2 (remember that $a^{ks1} = a^{ksk}p^{s1} = a^{ks}$)

Ignore the subset of states such that it must be $a^{ks} = 0$. Translating these restrictions on the vector a into matrix form, it must be

$$\begin{bmatrix} KI_I & Y(1)^{\setminus T}D\left(\ell_2^{\setminus}\right) & Y(2)^{\setminus T}D\left(\ell_2^{\setminus}\right) & & Y(K)^{\setminus T}D\left(\ell_2^{\setminus}\right) \\ C & I_{\hat{s}} + \Upsilon & I_{\hat{s}} & \cdots & I_{\hat{s}} \\ 0 & I_{\hat{s}} & 0 & \cdots & I_{\hat{s}} \\ & & & \ddots & \\ 0 & I_{\hat{s}} & I_{\hat{s}} & \cdots & 0 \end{bmatrix} a^T = 0,$$

where, now, $a = (a^{\Im}, ..., \left(a^{k1}, ..., a^{k\hat{s}}\right), ...)$, C is the $(\hat{s} \times I)$ matrix

$$\begin{bmatrix} c_1^{11}/p_1^{11} & \cdots & 0 \\ c_1^{12}/p^{21} & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix},$$

while Υ is the $(\hat{s} \times \hat{s})$ matrix

$$\begin{bmatrix} 0 & 0 & c_3^{11}/p^{11} & \cdots & c_{\hat{s}}^{11}/p^{11} \\ 0 & 0 & c_3^{12}/p^{21} & \cdots & c_{\hat{s}}^{12}/p^{21} \\ 0 & 0 & 0 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$Y^{\setminus}(k)$ is the (k-invariant) matrix of events k payoffs (in states 1,..., ŝ) and $D\left(\ell_2^{\setminus}\right)$ is the matrix having diagonal coefficients $\left(\ell_2^1, ..., \ell_2^{\hat{s}}\right)$. We need to show that this is true if and only if $a = 0$, i.e. that the matrix above has full rank. Use the first block of columns to get rid of all the assets payoffs but inside money and drop the corresponding rows and columns. Subtract the block of columns of k $= 1$ from the blocks k > 1, obtaining

$$\begin{bmatrix} K & [1]\,D\left(\ell_2^{\backslash}\right) & [0] & \cdots & [0] \\ \begin{bmatrix} c_1^{11}/p^{11} \\ c_1^{12}/p^{21} \end{bmatrix} & I_{\hat{s}} + \varUpsilon & -\varUpsilon & \cdots & -\varUpsilon \\ 0 & I_{\hat{s}} & -I_{\hat{s}} & \cdots & 0 \\ & & & \ddots & \\ 0 & I_{\hat{s}} & 0 & \cdots & -I_{\hat{s}} \end{bmatrix}.$$

Add the last $(K-1)$ blocks of columns to the block of k = 1, obtaining

$$\begin{bmatrix} K & \left[\ell_2^1 \cdots \ell_2^{\hat{s}}\right] & [0] & \cdots & [0] \\ \begin{bmatrix} c_1^{11}/p^{11} \\ c_1^{12}/p^{21} \\ \cdots \end{bmatrix} & I_{\hat{s}} - (K-2)\varUpsilon & -\varUpsilon & \cdots & -\varUpsilon \\ 0 & 0 & -I_{\hat{s}} & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & -I_{\hat{s}} \end{bmatrix}.$$

Taking into account the structure of the matrix $\varUpsilon$ (its first two columns are nil), the block 1 of columns has maximal rank. Moreover, multiply the first two columns of block 1 by c_1^{11}/p^{11} and c_1^{12}/p^{21}, respectively, and subtract from column i = 1, which then becomes $\left[\, K - \ell_2^1 c_1^{11}/p^{11} - \ell_2^2 c_1^{12}/p^{21} \;...\; 0 \,\right]^T$. Hence, the matrix above has full rank provided that $\left(K - \ell_2^1 c_1^{11}/p^{11} - \ell_2^2 c_1^{12}/p^{21}\right) \neq 0$.

Fix p^{11}, p^{21}, ℓ_1^1, ℓ_1^2, ℓ_H^1, ℓ_H^2 and K. By direct computation,
$\left(K - \ell_2^1 c_1^{11}/p^{11} - \ell_2^2 c_1^{12}/p^{21}\right) = \left(K - K\left((\ell_1^2 - \ell_H^2)\ell_2^1 + (\ell_H^1 - \ell_1^1)\ell_2^2\right)/(\ell_1^2\ell_H^1 - \ell_1^1\ell_H^2)\right).$

By assumption either $(\ell_1^2 - \ell_H^2) \neq 0$ or $(\ell_H^1 - \ell_1^1) \neq 0$ or both. If, for instance, the second term is non zero, perturb agent 2's utility function by $\delta_2 = \left[\delta_2^0, 0, \delta_2^2, ..., \delta_H^S\right]$, $\delta_2^2 \neq 0$, and δ_2^s, $s \neq 1, 2$, chosen so that $R(q)\delta_2 = 0$, which can always be done because $\left[\, y^{k1} \;\cdots\; y^{kS} \,\right]$ is in general position, for each k.

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